Analysis and design of Min-Sum-based decoders running on noisy hardware

Christiane Kameni-Ngassa*, Valentin Savin*, David Declercq#

* CEA-LETI, MINATEC campus, Grenoble, France
# ETIS ENSEA, Université de Cergy-Pontoise, France

Utilisation de codes détecteurs et/ou correcteurs d'erreurs pour fiabiliser les traitements numériques au sein de circuits non fiables

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Context & Objective

▪ Context
  ▪ Next-generation electronic circuit design
    ▪ increase in density integration
    ▪ process variations, post CMOS technologies
    ▪ lower power supply (reduction by 20% per technology node)
  ▪ Low energy consumption (sustainability concerns)
    ▪ aggressive voltage scaling

Reliability is among the ITRS Overall Design Technology Top-5 Challenges (2010)

▪ Objective
  ▪ Design fault tolerant solutions for LDPC decoders operating on circuits built out from unreliable (faulty) components
  ▪ Can MP decoders provide reliable error protection when they operate on faulty devices?
Min-Sum decoder on faulty devices

- Noisy components: new source of errors
  - Such errors may propagate through decoding iterations...
  - How does this impact on the error-correction capability of the decoder?
    - how to make sure that such an error propagation is not catastrophic?

- Theoretical analysis of “noisy” Min-Sum
  - Develop “noisy versions” of density-evolution
    - evaluate the theoretical performance loss due to noisy components
    - serve as guidelines for practical fault-tolerant implementations

- Practical fault-tolerant Min-Sum-based decoders
  - Evaluate the impact of faulty components on the performance of practical “finite-length” Min-Sum-based decoders
**Min-Sum decoder**

**Initialization:** \( \forall n = 1, ..., N; \forall m \in H(n) \)

\[
\gamma_n = \log(\Pr(x_n = 0 | y_n)/\Pr(x_n = 1 | y_n))
\]

\[
\alpha_{m,n} = \gamma_n
\]

**Iterations**

- **CNU:** \( \forall m = 1, ..., M; \forall n \in H(m) \)

\[
\beta_{m,n} = \left( \prod_{n' \in H(m) \setminus n} \text{sgn}(\alpha_{m,n'}) \right)_{n' \in H(m) \setminus n} \min_{n' \in H(m) \setminus n} (|\alpha_{m,n'}|)
\]

- **VNU:** \( \forall n = 1, ..., N; \forall m \in H(n) \)

\[
\alpha_{m,n} = \gamma_n + \sum_{m' \in H(n) \setminus m} \beta_{m',n}
\]

- **AP-LLR:** \( \forall n = 1, ..., N \)

\[
\tilde{\gamma}_n = \gamma_n + \sum_{m \in H(n)} \beta_{m,n}
\]
Min-Sum decoder on faulty devices

Initialization: \( \forall n = 1, \ldots, N; \forall m \in H(n) \)

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- **AP-LLR:** \( \forall n = 1, \ldots, N \)

\[ \tilde{\gamma}_n = \gamma_n + \sum_{m \in H(n)} \beta_{m,n} \]
Error models for faulty arithmetic units

- Probabilistic adder ($Q$ bits)
  - Two parameters: the **depth** $D$ and the **error probability** $P_a$
  - $P_a$ is the probability that an error occurs on at least one of the $D$ LSBs

<table>
<thead>
<tr>
<th>Correct Output</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error Pattern</td>
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<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
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</tbody>
</table>

- Probabilistic comparator
  - $P_c$ is the probability that the output is in error

Two's complement

rand integer in $[1, 2^D-1]$
Part I: Theoretical analysis of “noisy” Min-Sum decoder
Noisy density evolution

- Previous works
  - Varshney-2011
    - concentration and convergence properties were proved for the asymptotic performance of noisy message-passing decoders
    - density evolution equations were derived for the noisy Gallager-A decoder
  - Tabatabaei-2013
    - derived DE for noisy Gallager-B decoder defined over binary and non-binary alphabets
  - deal with very simple error models
    - emulate the noisy implementation of the decoder, by passing each of the exchanged messages through a binary (or non-binary) symmetric channel
Noisy density evolution

- We derived DE for fixed-point Min-Sum decoder
  - integrates above error models for arithmetic units (adder/comparator)

- Exchanged messages are random variables
  - Fixed-point implementation $\Rightarrow$ finite alphabet
  - $C$ the PMF of input LLR values $\gamma_n$ (depends only on the channel model)
  - $A^{(\ell)}, B^{(\ell)},$ and $\bar{C}^{(\ell)}$ the PMFs of $\alpha_{m,n}, \beta_{m,n},$ and $\tilde{\gamma}_n$ at iteration $\ell$

- DE equations (asymptotic performance)
  - Recursive formula (by tracking the update rules of exchanged messages):
    \[
    (A^{(\ell+1)}, B^{(\ell+1)}, \bar{C}^{(\ell+1)}) = f(A^{(\ell)}, B^{(\ell)}, \bar{C}^{(\ell)})
    \]
  - Under the assumption that incoming messages to any VNU and CNU are independent
  - In particular, the graph must be cycle-free
Noisy density evolution

- $P_\ell = \Pr(\tilde{Y}_n < 0)$ is the **error probability** at iteration $\ell$
- $P_\infty = \lim_{\ell \to \infty} P_\ell$ – **output error probability** (does not always exist!)

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**Useful decoder:** $P_\infty$ exits and $P_\infty < P_0$

**$\eta$-threshold:** $P_{th}(\eta) = \sup\{P_0 | P_\infty \text{ exists and } P_\infty < \eta\}$

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**DE equations** (asymptotic performance)

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Useful regions for Min-Sum decoder / BSC

- (3, 6)-regular LDPC codes, fixed-point MS
  - $Q = 5$ bits (number of bits of the adder)
  - $P_c = 0.001$ (error probability of the comparator)

Depth $D = 4$

Depth $D = 5$
Useful regions for Min-Sum decoder / BI-AWGN

\( D = 4, P_c = 0.001 \)

\( D = 4, P_c = 0.01 \)

\( D = 5, P_c = 0.001 \)

\( D = 5, P_c = 0.001 \)
First conclusion...

- Errors caused by noisy components do not necessarily propagate catastrophically through decoding iterations
  - Min-Sum decoder can still provide error protection with a given level of reliability, assuming that decoder’s components are reasonably noisy...

- Some characteristics of the Min-Sum decoder
  - Less sensitive to errors in comparators
  - Less sensitive to errors in the LSBs of the adder
  - Highly sensitive to errors in the sign bit of the adder
Part II:

Practical fault-tolerant Min-Sum-based decoders
Practical implementation of Min-Sum decoder

**Initialization:** \( \forall n = 1, \ldots, N; \forall m \in H(n) \)
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\gamma_n = \log(\Pr(x_n = 0 \mid y_n)/\Pr(x_n = 1 \mid y_n))
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\[
\alpha_{m,n} = \gamma_n
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**Iterations**

- **CNU:** \( \forall m = 1, \ldots, M; \forall n \in H(m) \)
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- **AP-LLR:** \( \forall n = 1, \ldots, N \)
\[
\tilde{\gamma}_n = \gamma_n + \sum_{m \in H(n)} \beta_{m,n}
\]

**Remark:** **MS(1)** and **MS(2)** are equivalent if exact (noiseless) arithmetic

**MS(1)** and **MS(2)** are **NOT equivalent** if probabilistic (noisy) arithmetic
Practical implementation of Min-Sum decoder

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  \[ \tilde{\gamma}_n = \gamma_n + \sum_{m \in H(n)} \beta_{m,n} \]

- **VNU**: \( \forall n = 1, \ldots, N; \forall m \in H(n) \)
  \[ \alpha_{m,n} = \tilde{\gamma}_n - \beta_{m,n} \]

The computation of \( \alpha_{m,n} \) takes \( d_n - 1 \) additions

(\( d_n \) denotes the degree of variable-node \( n \))

**Iterations**

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  \[ \alpha_{m,n} = \tilde{\gamma}_n - \beta_{m,n} \]

The computation of \( \alpha_{m,n} \) takes \( d_n + 1 \) additions

\[ \Rightarrow \] \( d_n \) additions to compute \( \tilde{\gamma}_n \), and 1 more addition to derive the \( \alpha_{m,n} \) value

\[ \Rightarrow \] An increased number of additions results in an increased error probability of \( \alpha_{m,n} \)
Practical implementation of Min-Sum decoder

- Mackay’s regular (3,6)-LDPC code, \([K = 504, N = 1008]\)
- Fixed-point MS decoder: 5 / 6 bits (exchanged messages / AP-LLR)

![Graph showing Frame Error Rate (FER) vs. Signal-to-Noise Ratio (SNR) for different depths and error probabilities.]

- Color code:
  - **Noiseless**
  - Depth = 3
  - Depth = 4
  - Depth = 5
  - Depth = 6

- Dashed curve: “DE-like” (1)
- Comp. err. prob: \(P_c = 0.01\)
- Adder err. prob: \(P_a = 0.01\)
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  - Depth = 5
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Dashed curve: “DE-like” (1)
Solid curve: Practical (2)
Performance of Min-Sum-based decoder

- Min-Sum-based decoders
  - improved versions of the MS algorithm, with only a very limited (usually negligible) increase in complexity
  - Offset-Min-Sum (OMS)
  - Self-Corrected Min-Sum (SCMS)
    - intrinsic ability to detect and discard unreliable messages during the iterative decoding process.

- Only “practical” implementations are considered
Performance of Min-Sum-based decoder

- Min-Sum-based decoders

- Self-Corrected Min-Sum (SCMS)
  - intrinsic ability to detect and discard unreliable messages during the iterative decoding process.

- a variable-to-check message \( \alpha_{m,n} \) is erased (set to zero) if its sign changed with respect to the previous iteration.

\[
\text{Initialization: } \forall n = 1, \ldots, N; \forall m \in H(n)
\]

\[
\begin{align*}
\gamma_n &= \log(\Pr(x_n = 0 \mid y_n)/\Pr(x_n = 1 \mid y_n)) \\
\alpha_{m,n} &= \gamma_n
\end{align*}
\]

\[
\text{Iterations}
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- \textbf{CNU: } \forall m = 1, \ldots, M; \forall n \in H(m)

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- \textbf{AP-LLR: } \forall n = 1, \ldots, N

\[
\tilde{\gamma}_n = \gamma_n + \sum_{m \in H(n)} \beta_{m,n}
\]

- \textbf{VNU: } \forall n = 1, \ldots, N; \forall m \in H(n)

\[
\alpha_{m,n}^{\text{tmp}} = \tilde{\gamma}_n - \beta_{m,n}
\]

- if \( \text{sgn}(\alpha_{m,n}^{\text{tmp}}) = \text{sgn}(\alpha_{m,n}) \) then \( \alpha_{m,n} = \alpha_{m,n}^{\text{tmp}} \)
- else \( \alpha_{m,n} = 0 \)
Performance of Min-Sum-based decoder

- Mackay’s regular (3,6)-LDPC code, \([K = 504, N = 1008]\)
- Fixed-point decoders: 4 / 5 bits (exchanged messages / AP-LLR)

![Graph showing performance curves for different codes and error rates]

- Comp. err. prob: \(P_c = 0.01\)
- Adder err. prob: \(P_a = 0.01\)
- Depth = 4

Color code:
- SCMS
- MS
- OMS

Dashed curves: noiseless
Solid curves: noisy

Remark: noiseless SCMS achieves \(\approx\) Belief Propagation performance
Performance of Min-Sum-based decoder

- Mackay’s regular (3,6)-LDPC code, \([K = 504, N = 1008]\)
- Fixed-point decoders: 4 / 5 bits (exchanged messages / AP-LLR)

Requested channel \(E_b/N_0\) for target decoded BER = \(10^{-5}\)

\((E_b/N_0 = \infty \text{ means target BER cannot be achieved})\)

- Error prob. of the adder (depth = 4)
- Error prob. of the comparator
Conclusion

- “Adjustable” error-models for noisy Min-Sum-based decoders
- Density evolution analysis of the noisy Min-Sum decoder
  - proved that error protection (with a certain level of reliability) is still possible
  - characterized the sensitivity of the decoder to variations of the parameters of the error model, in terms of useful regions
- Finite-length performance of Min-Sum-based decoders
  - highlighted the limitations of the theoretical analysis with respect to practical implementations
  - evaluate finite-length performance for various parameters of the hardware noise model
  - **SCMS**: intrinsic ability to detect and discard unreliable messages, which proves to be particularly useful for noisy implementations
Merci de votre attention