Orthonormal Basis Functions For Modeling Continuous-time Fractional Systems

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SDNE – October 16th 2003
Also presented in SySId'03

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Objective

Synthesis of fractional orthogonal basis for system identification purposes

Extending Laguerre function’s definition to fractional orders

Outline

- Definitions and generalities
- Synthesis of orthogonal basis function using Gramm-Schmidt procedure
- Completeness of the basis
- Identification using orthogonal basis
- Conclusions and actual work

Definitions and generalities

Fractional systems


u(t)

Fractional system

y(t)

Differentiation equation

$$y(t)+a_1\left(\frac{d}{dt}\right)^{n_{a_1}} y(t)+\ldots+a_L\left(\frac{d}{dt}\right)^{n_{a_L}} y(t)=b_0\left(\frac{d}{dt}\right)^{n_{b_0}} u(t)+\ldots+b_M\left(\frac{d}{dt}\right)^{n_{b_M}} u(t)$$

$n_{a_1}, n_{a_2}, \ldots, n_{a_L}, n_{b_0}, n_{b_2}, \ldots, n_{b_M}$ are real

Fractional differentiation (Riemann Liouville, XIXème century)

$$D^n f(t) \overset{\Delta}{=} \frac{1}{\Gamma(m-n)}\left(\frac{d}{dt}\right)^m \left(\int_0^t \frac{f(\tau)}{(t-\tau)^{1-(m-n)}} d\tau\right) \quad \text{where} \quad \begin{cases} t>0 \\ \mathcal{Re}(n)>0 \\ m=\lfloor \mathcal{Re}(n) \rfloor+1 \end{cases}$$

Representation of fractional systems

Differentiation equation

$$y(t)+a_1\left(\frac{d}{dt}\right)^{n_{a_1}}y(t)+\ldots+a_L\left(\frac{d}{dt}\right)^{n_{a_L}}y(t)=b_0\left(\frac{d}{dt}\right)^{n_{b_0}}u(t)+\ldots+b_M\left(\frac{d}{dt}\right)^{n_{b_M}}u(t)$$

$n_{a_1}, n_{a_2}, \ldots., n_{a_L}, n_{b_0}, n_{b_2}, \ldots., n_{b_M}$ are real

Transfer function

$$H(s)=\frac{\mathscr{L}\{y(t)\}}{\mathscr{L}\{u(t)\}}=\frac{b_0 s^{n_{b_0}}+\ldots+b_M s^{n_{b_M}}}{1+a_1 s^{n_{a_1}}+\ldots+a_L s^{n_{a_L}}}$$

Main characteristic of fractional systems

Example

The impulse response of: $\dfrac{1}{\boldsymbol{s^n} + \boldsymbol{\lambda_l}}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^n - \lambda_l}\right) = \underbrace{\sum_{k=1}^{\text{poles number}} \frac{s_k}{n\lambda_l} \mathrm{e}^{ts_k}}_{\text{exponentials modes}} + \underbrace{\frac{\sin(n\pi)}{\pi} \int_0^{\infty} \frac{x^n \mathrm{e}^{-tx}}{x^{2n} - 2\lambda_l x^n \cos(n\pi) + \lambda_l^{\,2}} dx}_{\text{aperiodic modes}}$$

Orthogonal functions

⌦ Orthogonality

Plancherel's theorem

$$\left\langle g_n , g_m \right\rangle = \int_0^{\infty} g_n(t) g_m(t)\, dt = \delta_{nm} \quad \Longleftrightarrow \quad \left\langle G_n , G_m \right\rangle = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} G_n(j\omega) \overline{G_m(j\omega)}\, d\omega = \delta_{nm}$$

⌦ Mean square error

$$\left\| e \right\|^2 = \left\langle e, e \right\rangle = \int_0^{\infty} e(t) e(t)\, dt$$

⌦ In a context of system approximation

$$e(t) = f(t) - \sum_{i=1}^{N} a_i g_i(t) \qquad \underline{A} = \underbrace{\cancel{\left\langle \underline{g}, \underline{g}^T \right\rangle^{-1}}}_{I} \left\langle \underline{g}, f \right\rangle$$

Most common orthogonal functions in control

- ⌧ **Laguerre functions:** Unique pole

  Well suited for modeling systems with a single predominant dynamical behavior

- ⌧ **Kautz functions:** Two complex conjugate poles

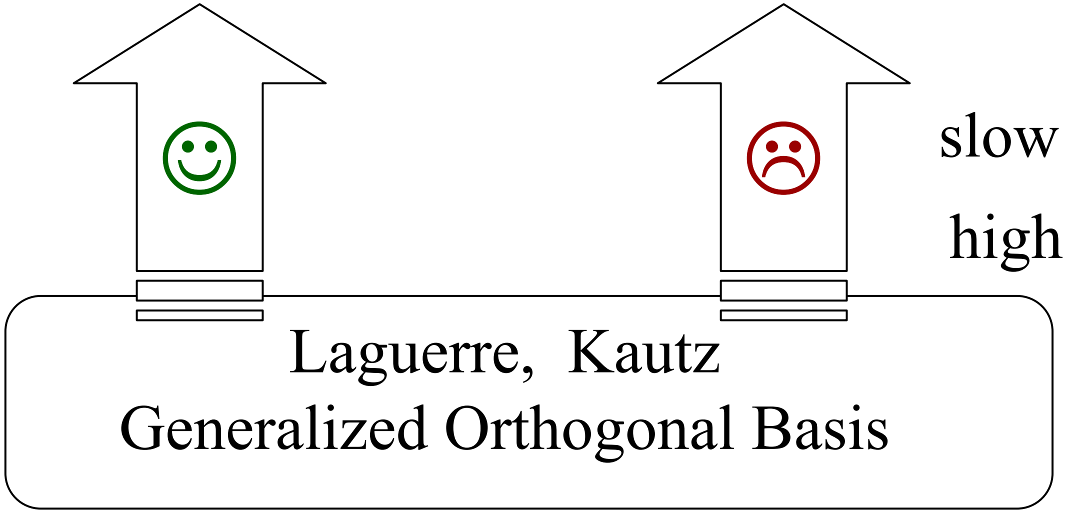
  Well suited for modeling systems having predominant oscillatory behavior

- ⌧ **GOB functions:** Any number of poles

  Extend the two former definitions to any number of real or complex conjugate poles

Classical orthonormal basis

$$\mathscr{L}^{-1}\left(\frac{1}{s^{n}-\lambda_{l}}\right)=\underbrace{\sum_{k=1}^{\text{poles number}}\frac{s_{k}}{n\lambda_{l}}\,\mathrm{e}^{ts_{k}}}_{\text{exponentials modes}}+\underbrace{\frac{\sin(n\pi)}{\pi}\int_{0}^{\infty}\frac{x^{n}\mathrm{e}^{-tx}}{x^{2n}-2\lambda_{l}x^{n}\cos(n\pi)+\lambda_{l}^{2}}\,dx}_{\text{aperiodic modes}}$$



slow convergence

high basis length

Laguerre, Kautz
Generalized Orthogonal Basis

functions with integer differentiation

Laguerre basis

- ⌧ *n* integer

$$l_n(t) = \sqrt{2\lambda}\,\frac{e^{\lambda t}}{n!}\,\frac{d^n\left(t^n e^{-2\lambda t}\right)}{dt^n}$$

$$L_n(s) = \sqrt{2\lambda}\,\frac{\left(s-\lambda\right)^n}{\left(s+\lambda\right)^{n+1}}$$

- ⌧ as soon as *n* is non-integer

$$\int_0^\infty \left(l_n(t)\right)^2 dt = \infty \qquad \forall\, n \in \mathbb{R}^+ - \mathbb{N}$$

Abbott P.C. (2000)

Laguerre basis

- **Classical Laguerre functions**

$$L_n(s) = \sqrt{2\lambda}\,\frac{(s-\lambda)^n}{(s+\lambda)^{n+1}}$$

- **can be used to approximate fractional systems *f*(*t*) iff:**

$$\int_0^\infty \left(f(t)\right)^2 dt < \infty$$

- **Caution! a stable fractional system can have:**

$$\int_0^\infty \left(f(t)\right)^2 dt = \infty$$

See Malti et al. (ASME'2003)

Laguerre basis

☒ When the fractional systems is $L_2[0, \infty[$

$$F(s) \approx \sum_{n=0}^{\boxed{N}} L_n(s)$$

Has to be large

☒ because:

- Fails to capture the aperiodic multimode

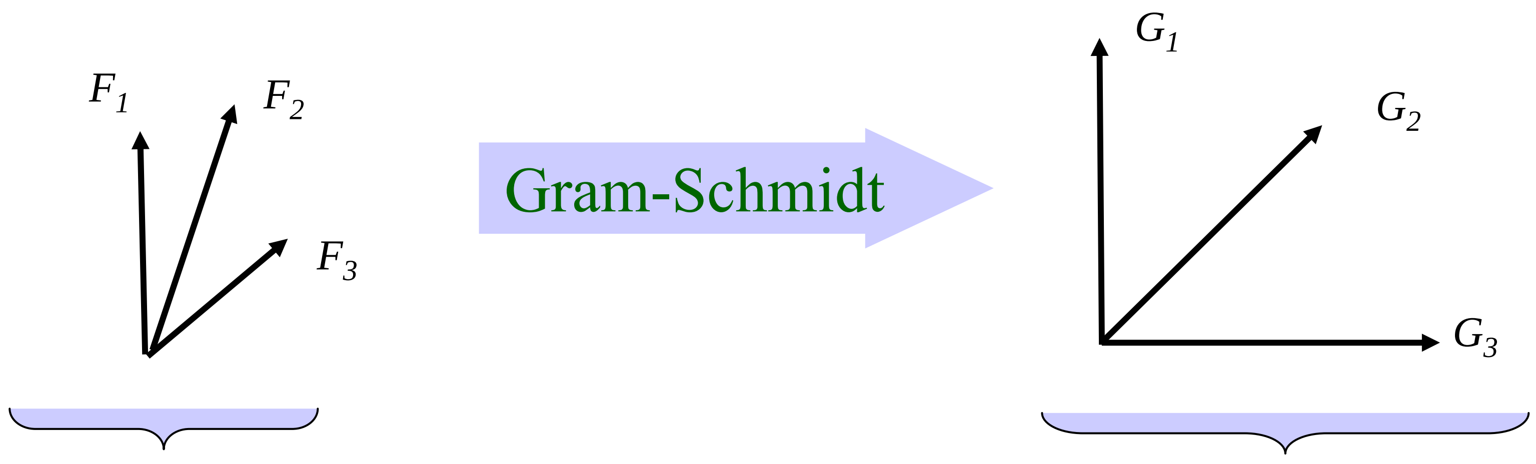
☒ Idea: why not synthesizing a fractional orthogonal basis ?

Synthesis of a new fractional orthogonal basis

The new basis will have two tuning parameters:

- Commensurate order *n*
- Eigenvalue λ

Synthesis of new basis



Gram-Schmidt

Generating functions

Orthonormal basis

Generating functions

$$F_1(s)=\frac{1}{s^n+\lambda},\ F_2(s)=\frac{1}{\left(s^n+\lambda\right)^2},\dots,F_m(s)=\frac{1}{\left(s^n+\lambda\right)^m},\dots \qquad m=1,2,3\dots$$

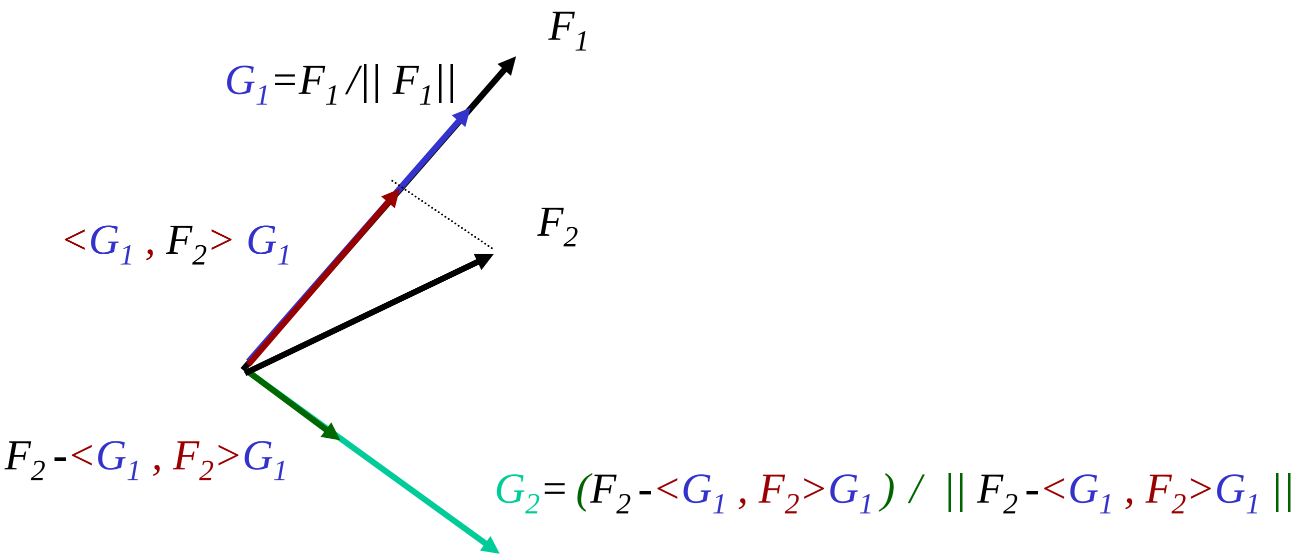
$$n>0$$

$$\lambda>0$$

n and λ are the same for all the functions

Orthogonalisation Procedure

Gram-Schmidt procedure



F_1

$G_1 = F_1 / \| F_1 \|$

F_2

$<G_1 , F_2> G_1$

$F_2 - <G_1 , F_2> G_1$

$G_2 = (F_2 - <G_1 , F_2> G_1) / \| F_2 - <G_1 , F_2> G_1 \|$

1st basis function

$$G_1(s) = \alpha_{1,1} \frac{1}{s^n + \lambda}$$

$$\alpha_{1,1} = \left\| \frac{1}{s^n + \lambda} \right\|^{-1}$$

2nd basis function

$$G_2(s) = \alpha_{2,1} G_1(s) + \alpha_{2,2} F_2(s)$$

$$\langle G_1(s), G_2(s) \rangle = 0$$

$$\langle G_2(s), G_2(s) \rangle = 1$$

Orthogonalisation Procedure

Gram-Schmidt procedure

$$G_1(s) = \alpha_{1,1} F_1(s)$$

$$G_2(s) = \alpha_{2,1} F_1(s) + \alpha_{2,2} F_2(s)$$

Matrix notation

$$\begin{bmatrix} G_1(s) \\ G_2(s) \\ \vdots \\ G_{M-1}(s) \\ G_M(s) \end{bmatrix} = \begin{bmatrix} \alpha_{1,1} & 0 & \cdots & 0 & 0 \\ \alpha_{2,1} & \alpha_{2,2} & & & 0 \\ & \vdots & \ddots & & \vdots \\ \alpha_{M-1,1} & \alpha_{M-1,2} & \cdots & \alpha_{M-1,M-1} & 0 \\ \alpha_{M,1} & \alpha_{M,2} & \cdots & \alpha_{M,M-1} & \alpha_{M,M} \end{bmatrix} \begin{bmatrix} F_1(s) \\ F_2(s) \\ \vdots \\ F_{M-1}(s) \\ F_M(s) \end{bmatrix} \quad \Rightarrow \quad \underline{\mathbf{G}}(s) = \underline{\underline{\boldsymbol{\alpha}}}\, \underline{\mathbf{F}}(s)$$

Orthogonalisation

$$\left\langle \underline{\mathbf{G}}, \underline{\mathbf{G}}^T \right\rangle = I \quad \Rightarrow \quad \underline{\underline{\boldsymbol{\alpha}}} \left\langle \underline{\mathbf{F}}, \underline{\mathbf{F}}^T \right\rangle \underline{\underline{\boldsymbol{\alpha}}}^T = I \quad \Rightarrow \quad \underline{\underline{\boldsymbol{\alpha}}} = \text{Cholesky}\left(\left\langle \underline{\mathbf{F}}, \underline{\mathbf{F}}^T \right\rangle^{-1} \right)$$

Orthogonalisation Procedure

Difficulty – how to compute : $\left\langle F_{m_1}, F_{m_2} \right\rangle$

When differentiation order is non-integer

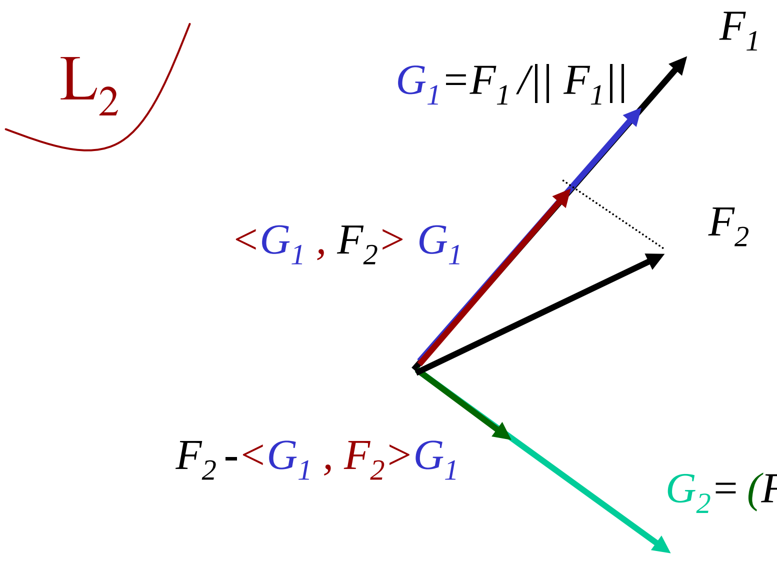
- F_{m1} and F_{m2} are not holomorphic functions
- a plan cut is necessary between 0 and ∞
- the scalar product is computed along a Bromwich-Wagner contour

The difficulty is to compute:

$$\left\langle F_{m_1}, F_{m_2} \right\rangle = \frac{1}{2\pi j} \int_{-j\omega}^{j\infty} F_{m_1}(j\omega)\overline{F_{m_2}(j\omega)}\, d\omega$$

Necessary condition

Gram-Schmidt procedure



F_1

$G_1 = F_1 / \| F_1 \|$

F_2

$\langle G_1, F_2 \rangle G_1$

$F_2 - \langle G_1, F_2 \rangle G_1$

$G_2 = (F_2 - \langle G_1, F_2 \rangle G_1) / \| F_2 - \langle G_1, F_2 \rangle G_1 \|$

$$\|F_1\| < \infty, \|F_2\| < \infty, \|F_3\| < \infty, \ldots.$$

$$\|F_1(s)\| = \left\| \frac{1}{s^n + \lambda} \right\| < \infty \Leftrightarrow 0.5 < n < 2$$

1st basis function

$$G_1(s) = \alpha_{1,1} \frac{1}{s^n + \lambda}$$

$$\alpha_{1,1} = \left\| \frac{1}{s^n + \lambda} \right\|^{-1}$$

necessary condition
$0.5 < n < 2$

Example 1

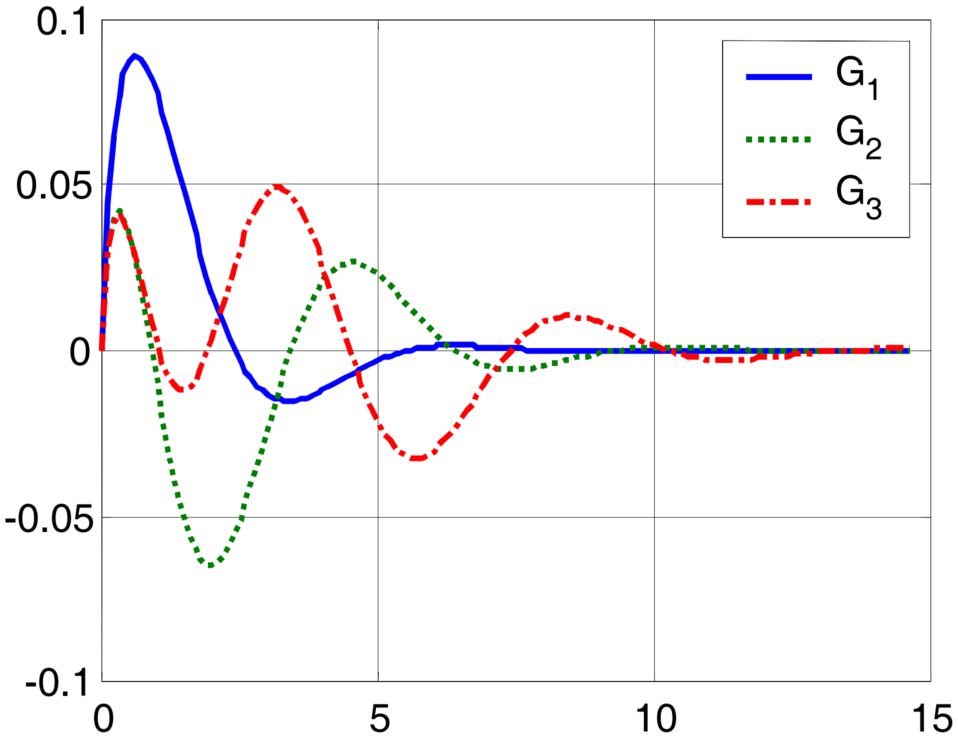
$n = 1.5$, $\lambda = 1.5$

$$F_1(s) = \frac{1}{s^{1.5}+1.5}, F_2(s) = \frac{1}{\left(s^{1.5}+1.5\right)^2}, F_3(s) = \frac{1}{\left(s^{1.5}+1.5\right)^3}$$

$$G_1(s) = \frac{1.49}{s^{1.5}+1.5}$$

$$G_2(s) = \frac{0.7}{s^{1.5}+1.5} + \frac{-2.37}{\left(s^{1.5}+1.5\right)^2}$$

$$G_3(s) = \frac{-0.12}{s^{1.5}+1.5} + \frac{-1.17}{\left(s^{1.5}+1.5\right)^2} + \frac{3.56}{\left(s^{1.5}+1.5\right)^3}$$



Example 2

$n = 1, \lambda = 1.5$

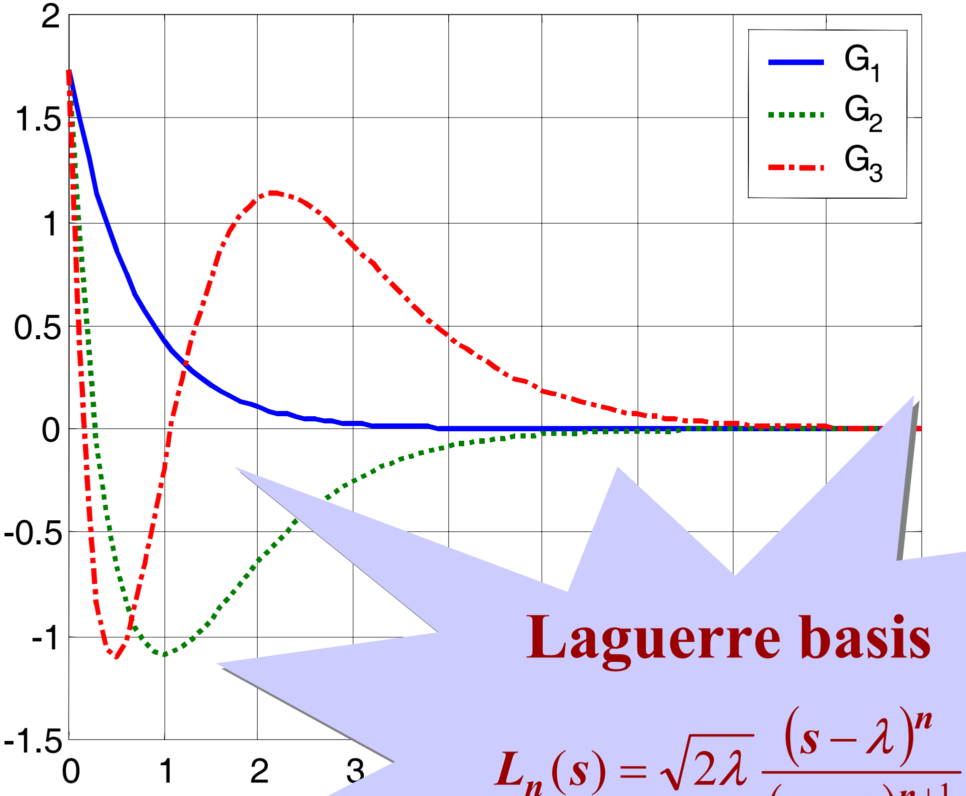
$$F_1(s) = \frac{1}{s^1 + 1.5}, F_2(s) = \frac{1}{\left(s^1 + 1.5\right)^2}, F_3(s) = \frac{1}{\left(s^1 + 1.5\right)^3}$$

$$G_1(s) = \frac{\sqrt{3}}{s + 1.5}$$

$$G_2(s) = \sqrt{3}\left(\frac{1}{s + 1.5} + \frac{-3}{(s + 1.5)^2}\right) = \sqrt{3}\,\frac{(s - 1.5)^2}{(s + 1.5)^3}$$

$$G_3(s) = \sqrt{3}\left(\frac{1}{s + 1.5} + \frac{-6}{(s + 1.5)^2} + \frac{9}{(s + 1.5)^3}\right)$$

$$= \sqrt{3}\,\frac{(s - 1.5)^3}{(s + 1.5)^4}$$

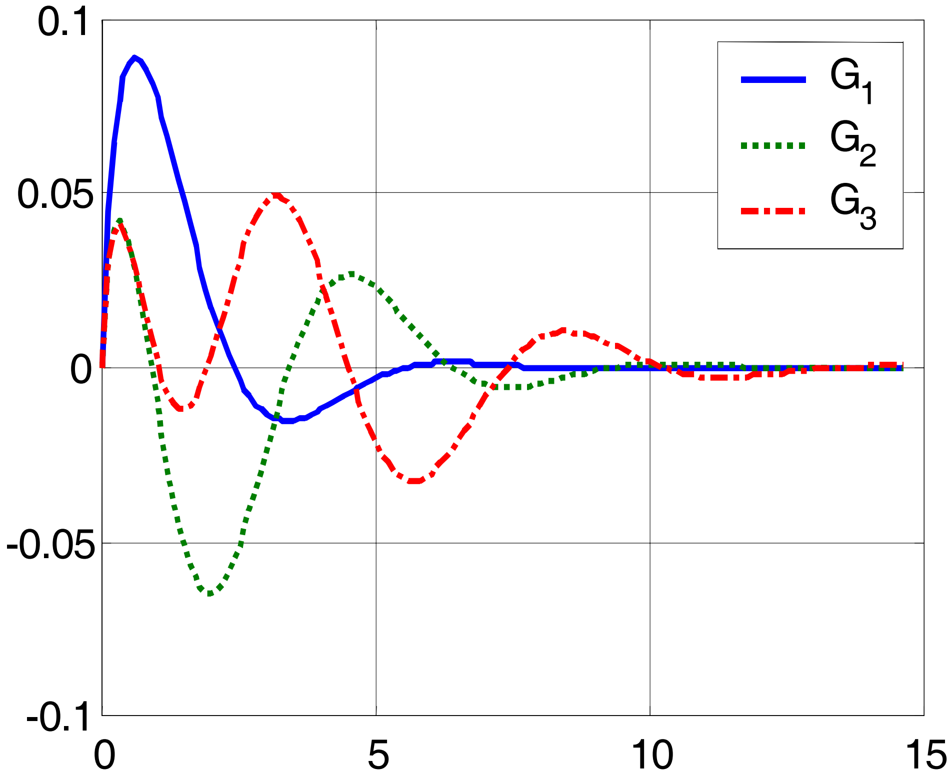


Laguerre basis

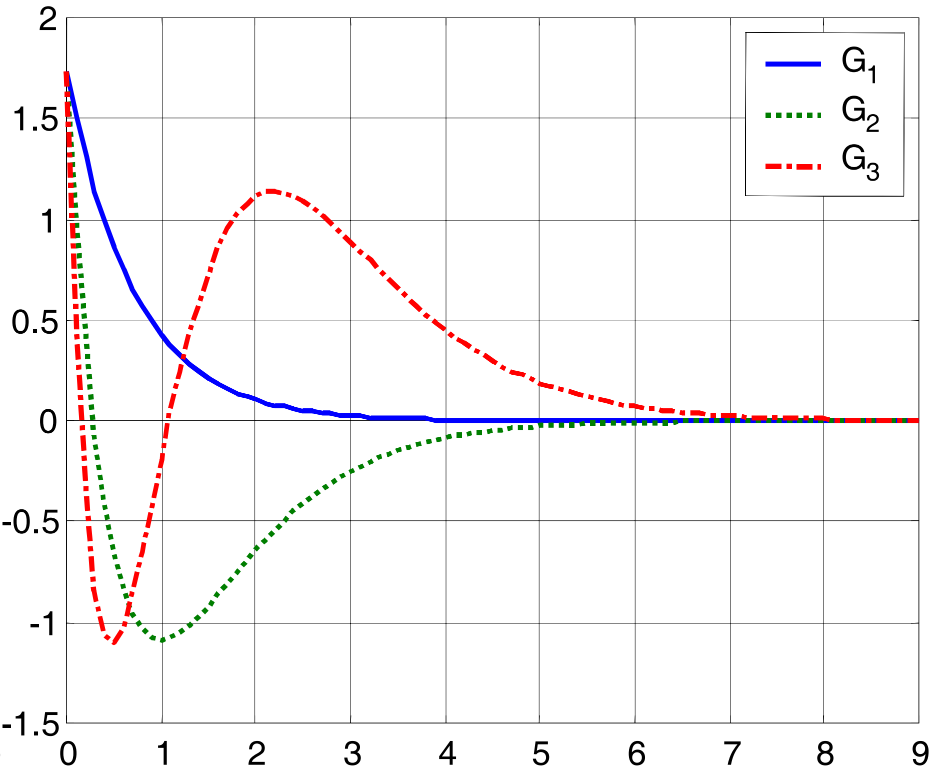
$$L_n(s) = \sqrt{2\lambda}\,\frac{(s - \lambda)^n}{(s + \lambda)^{n+1}}$$

Fractional order effect (1)

λ=1.5



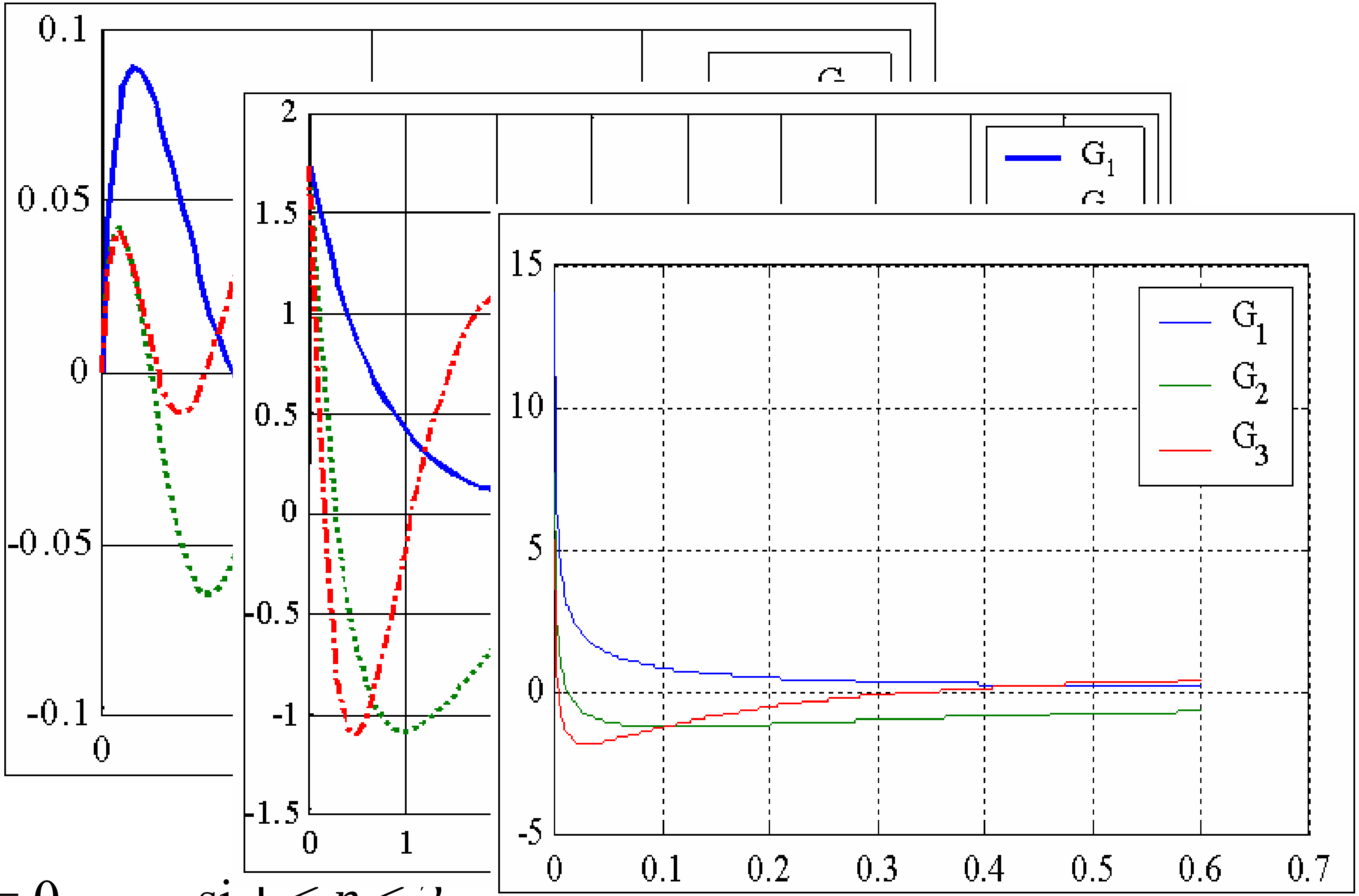
$n=1{,}5$



$n=1$

n **↗ ⟺ Oscillations ↗**

Fractional order effect $_{(2)}$



- ➢ $g_m(0) = 0$ si $1 < n < 2$
- ➢ $g_m(0) = (2\lambda)^{0.5}$ si $n = 1$ (Laguerre functions)
- ➢ $g_m(0) = \infty$ si $\frac{1}{2} < n < 1$

Completeness of the new basis

If the basis is complete then:

$$\forall f(t) \in L_2[0, \infty[\;\Rightarrow\; f(t) = \sum_{n=0}^{\infty} a_n g_n(t)$$

Completeness

Theorem

$$F_1(s)=\frac{1}{\left(s^n+\lambda\right)^1},F_2(s)=\frac{1}{\left(s^n+\lambda\right)^2},\ldots,F_m(s)=\frac{1}{\left(s^n+\lambda\right)^m},\ldots$$

The proposed orthogonal basis spans completely $L_2[0, \infty[$ iff
$0.5 < n < 2$

Extension to any commensurate order 0 < *n* < 2

The proposed orthogonal basis spans completely $L_2[0, \infty[$, for every
$0 < n < 2$, iff :

$$F_1(s)=\frac{1}{\left(s^n+\lambda\right)^k},F_2(s)=\frac{1}{\left(s^n+\lambda\right)^{k+1}},\ldots,F_m(s)=\frac{1}{\left(s^n+\lambda\right)^m},\ldots$$

$$k > \left\lfloor \frac{1}{2n} \right\rfloor, \left\lfloor \frac{1}{2n} \right\rfloor + 1, \cdots, \infty$$

Particular case Laguerre functions

Completeness theorem

Laguerre functions are known to span $L_2[0, \infty[$

Particular case of our theorem

Indeed, if $n = 1$

$$k > \left\lfloor \frac{1}{2 \times 1} \right\rfloor = \lfloor 0.5 \rfloor = 1$$

$$F_1(s) = \frac{1}{\left(s^n + \lambda\right)^1}, F_2(s) = \frac{1}{\left(s^n + \lambda\right)^2}, \ldots, F_m(s) = \frac{1}{\left(s^n + \lambda\right)^m}, \ldots$$

Extension to any commensurate order $0 < n < 2$

Take the generating functions:

$$F_1(s) = \frac{1}{\left(s^n + \lambda\right)^k}, F_2(s) = \frac{1}{\left(s^n + \lambda\right)^{k+1}}, \ldots, F_m(s) = \frac{1}{\left(s^n + \lambda\right)^m}, \ldots$$

Set, for example, commensurate order to 0.21, then:

$$k > \left\lfloor \frac{1}{2 \times 0.21} \right\rfloor = \lfloor 2.38 \rfloor \Rightarrow k = 3$$

Hence the generating functions are:

$$F_1(s) = \frac{1}{\left(s^{0.21} + \lambda\right)^3}, F_2(s) = \frac{1}{\left(s^{0.21} + \lambda\right)^4}, \ldots, F_m(s) = \frac{1}{\left(s^{0.21} + \lambda\right)^m}, \ldots$$

System identification using fractional orthogonal basis

System identification - Principe

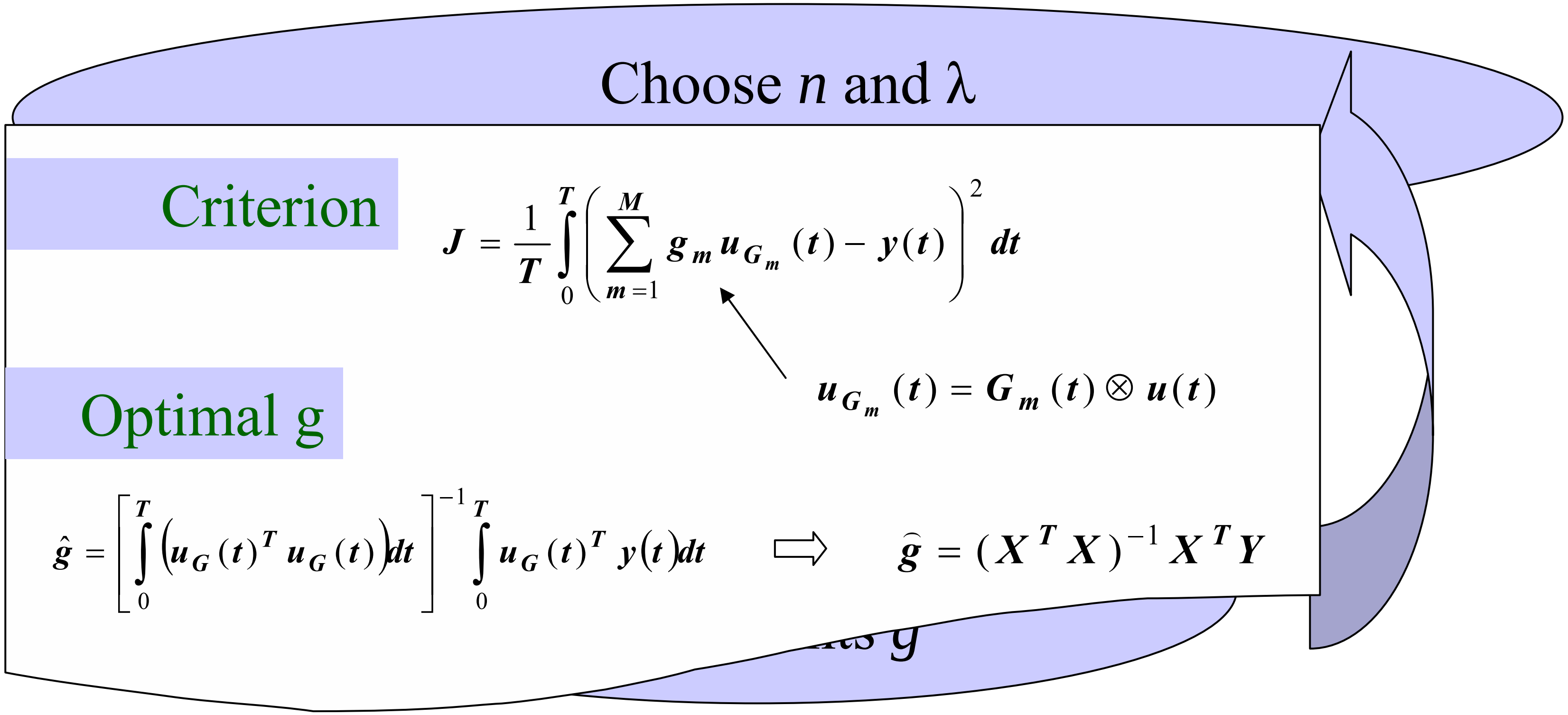
Model structure

$$H(s) \approx \sum_{m=1}^{M} g_m G_m(s)$$

$$H(s) \approx \underbrace{[g_1\ \ g_{2,}\ldots g_M]}_{g} * \left[G_1(s)\ G_2(s),\cdots G_M(s)\right]^T$$

Wahlberg 91

Algorithm



Choose *n* and λ

Criterion

$$J = \frac{1}{T}\int_0^T \left(\sum_{m=1}^{M} g_m u_{G_m}(t) - y(t)\right)^2 dt$$

$$u_{G_m}(t) = G_m(t) \otimes u(t)$$

Optimal g

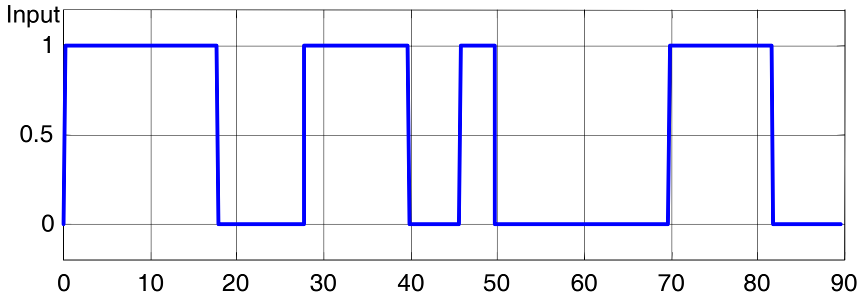
$$\hat{g} = \left[\int_0^T \left(u_G(t)^T u_G(t)\right)dt\right]^{-1} \int_0^T u_G(t)^T y(t)dt \quad \Rightarrow \quad \hat{g} = (X^T X)^{-1} X^T Y$$

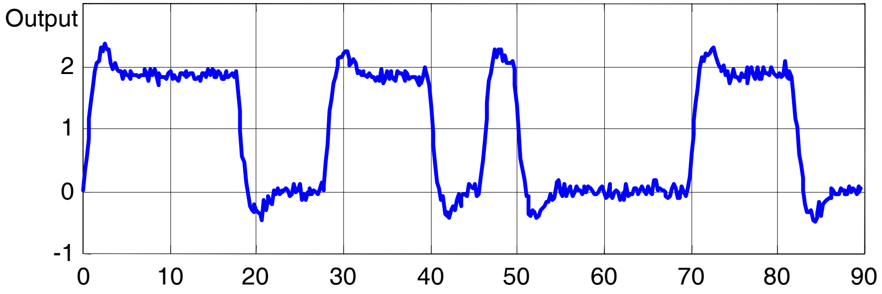
Academic example

Initial system

$$H(s)=\frac{1}{s^{1.5}+1.5}+\frac{1}{s^{1.5}+1.6}+\frac{1}{s^{1.5}+1.8}$$

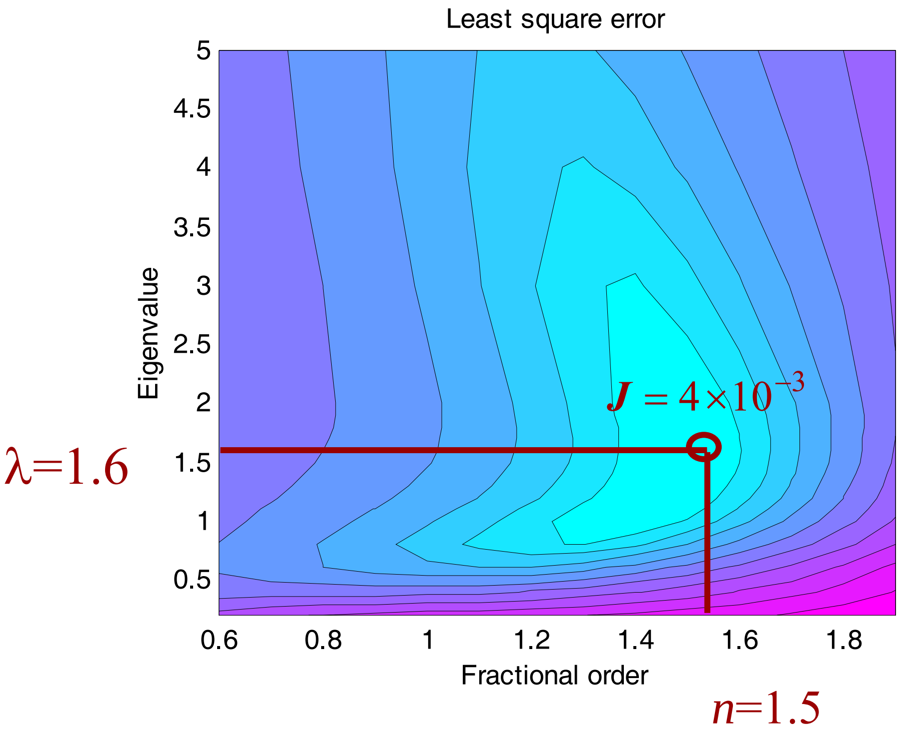
Identification data





Optimisation

$$\boldsymbol{H}(\boldsymbol{s}) = \frac{1}{\boldsymbol{s}^{1.5}+1.5} + \frac{1}{\boldsymbol{s}^{1.5}+1.6} + \frac{1}{\boldsymbol{s}^{1.5}+1.8}$$



$\boldsymbol{J} = 4\times 10^{-3}$

λ=1.6

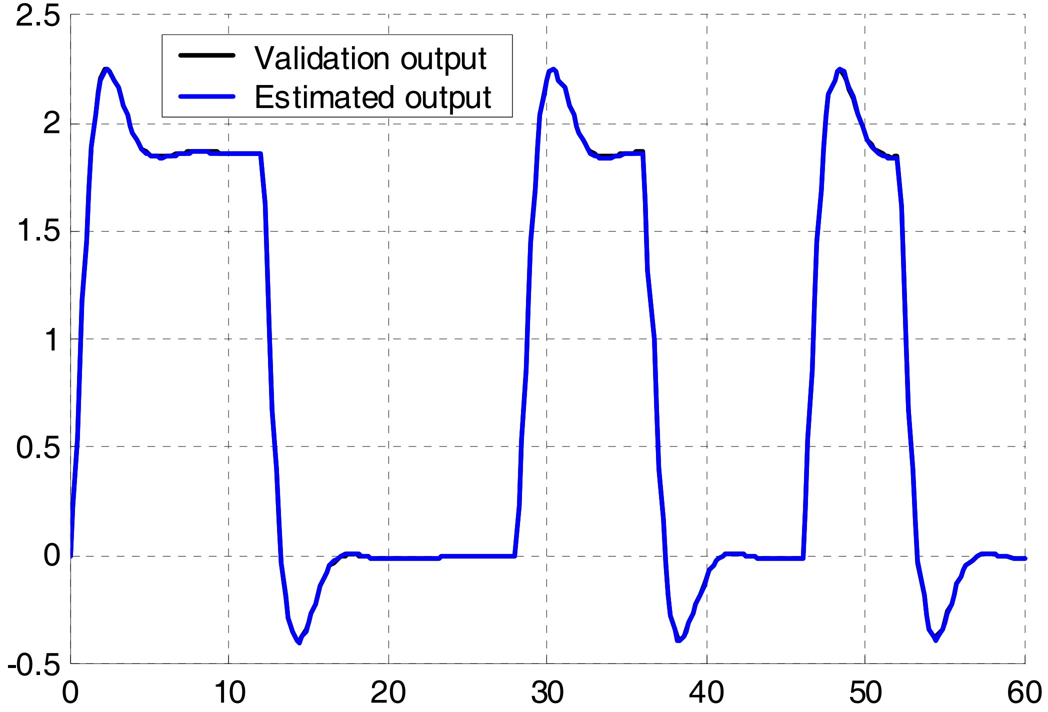
n=1.5

$$\widehat{\boldsymbol{H}}(\boldsymbol{s}) = 1.89\,\boldsymbol{G}_1(\boldsymbol{s}) - 0.01\boldsymbol{G}_2(\boldsymbol{s})$$

$$\hat{\boldsymbol{H}}(\boldsymbol{s}) \cong 1.48\,\boldsymbol{L}_1(\boldsymbol{s}) - 0.94\,\boldsymbol{L}_2(\boldsymbol{s}) - 0.54\,\boldsymbol{L}_3(\boldsymbol{s}) + 0.44\,\boldsymbol{L}_4(\boldsymbol{s}) + 0.22\,\boldsymbol{L}_5(\boldsymbol{s})$$

Validation

Validation data



Conclusions

A new fractional orthonormal basis is synthesized. It has two tuning parameters:

commensurate order n and eigenvalue λ

Most propriety used in classical identification algorithms are still valid ➔ two tuning parameters are to be set (instead of one as in Laguerre case)

The new basis can be used in model reduction

Recent results

The completeness of the fractional basis is proved. The completeness condition is determined. (yet to be published).

The fractional order interval is extended to]0, 2[.

Kautz and GOB functions are currently being generalized
A paper will be proposed in FDA’04

Problem at stake

Let:

$$F(s) = \frac{\sum_{m=0}^{m_A} a_m s^{\alpha_m}}{1 + \sum_{m=1}^{m_B} b_m s^{\beta_m}}$$

Although *F*(*s*) is stable, it has an ∞ L_2 norm if:

$$\beta_{mB} \leq \alpha_{mA} + \frac{1}{2}$$

See Malti et al. (ASME'2003)

This particular class of fractional models cannot be approximated by orthogonal functions because …

… the approximation is based on the minimization of the the MSE which indeed is ∞.