# Necessary Conditions for the Existence of Global Solutions to Systems of Fractional Differential Equations

S. KERBAL, Department of Mathematics,
Qaboos University, Sultanate of Oman

M. KIRANE, Laboratoire de Mathématiques
et Applications, Université de La Rochelle

*Necessary conditions for the existence of global solutions to a $2 \times 2$-system of frac-diff equations are presented.*
*Our method of proof relies on a choice of the test function in the weak formulation of the solutions to the system.*

# 1. INTRODUCTION

**Aim**: Exponents threshold for the existence of global solutions to the system (for $t > 0$)

$$\gamma\, u'(t) + \delta\, D^{\alpha}_{0|t}(u - u_0) = f(t)\, |v|^q + F(t), \quad (1)$$

$$\lambda\, v'(t) + \mu\, D^{\beta}_{0|t}(v - v_0) = g(t)\, |u|^p + G(t), \quad (2)$$

subject to the initial conditions

$$u(0) = u_0, \qquad v(0) = v_0, \qquad (3)$$

where $1 < p, q,\ 0 < \alpha, \beta \leq 1, \gamma, \delta, \lambda, \mu$ are constants, the functions $f(t), g(t)$ are positive while $F(t)$ and $G(t)$ are given functions with nonnegative averages.

For $0 < \theta < 1$, $D^{\theta}_{0|t}$ stands for the time-fractional derivative in the Riemann-Liouville sense defined by

$$(D^{\theta}_{0|t}f)(t) = \frac{1}{\Gamma(1-\theta)}\frac{d}{dt}\int_0^t \frac{f(\sigma)}{(t-\sigma)^{\theta}}\,d\sigma.$$

This fractional derivative is said to be *left-handed*.

The *right-handed* fractional derivative is defined by

$$(D^{\theta}_{t|T}f)(t) = \frac{-1}{\Gamma(1-\theta)}\frac{d}{dt}\int_t^T \frac{f(\sigma)}{(\sigma-t)^{\theta}}\,d\sigma.$$

The Caputo derivative

$$(\mathbf{D}^{\alpha}_{0|t}\Psi)(t) = -\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\Psi'(\sigma)}{(t-\sigma)^{\alpha}}\,d\sigma,$$

is related to the Riemann-Liouville derivative by

$$\mathbf{D}^{\alpha}_{0|t}\Psi(t) = D^{\alpha}_{0|t}[\Psi(t)-\Psi(0)].$$

The formula of integration by parts:

$$\int_0^T f(t)\,(\mathbf{D}^{\alpha}_{0|t}g)(t)\,dt = \int_0^T (\mathbf{D}^{\alpha}_{t|T}f)(t)\,g(t)\,dt$$

where

$$(\mathbf{D}^{\alpha}_{0|t}g)(t) = \frac{1}{\Gamma(1-\alpha)}\Big[\frac{g(0)}{t^{\alpha}} + \int_0^t \frac{g'(\sigma)}{(t-\sigma)^{\alpha}}\,d\sigma\Big],$$

$$(\mathbf{D}^{\alpha}_{t|T}f)(t) = \frac{1}{\Gamma(1-\alpha)}\Big[\frac{f(T)}{(T-t)^{\alpha}} - \int_t^T \frac{f'(\sigma)}{(\sigma-t)^{\alpha}}\,d\sigma\Big].$$

References consulted: (Local/ global existence, large time behavior, blowing-up and estimates of the interval of existence)

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Local existence and global existence under natural conditions (sub-linear growth of the nonlinearities): Kilbas, Trujillo, Michalski, Diethelm and Ford.

Blow-up and estimate of the time interval: Arias, Castillo, Bushell, and W. Okrasinski.

## Particular systems.

$$\begin{cases} u'(t) = |v|^q, & t > 0, \\ v'(t) = |u|^p, & t > 0, \end{cases}$$

For $pq > 1$ its solutions blow-up in finite time:

$$u(t) = C_1(T_0 - t)^{-(q+1)/(pq-1)},$$

$$v(t) = C_2(T_0 - t)^{-(p+1)/(pq-1)},$$

$$0 < t < T_0 < \infty,$$

where

$$C_1 = \left[(p+1)^q(q+1)/(pq-1)^{q+1}\right]^{1/(pq-1)},$$

$$C_2 = \left[(p+1)(q+1)^p/(pq-1)^{p+1}\right]^{1/(pq-1)}.$$

The blowing-up solutions that satisfy the other particular system containing only fractional derivatives

$$\begin{cases} D^{\alpha}_{0|t}(u-u_0) &= |v|^q, \quad t>0, \\ D^{\beta}_{0|t}(v-v_0) &= |u|^p, \quad t>0, \end{cases}$$

has the form

$$u(t) = \tilde{C}_1 (T_0 - t)^{-(\alpha+q\beta)/(pq-1)},$$
$$v(t) = \tilde{C}_2 (T_0 - t)^{-(\beta+p\alpha)/(pq-1)}$$

$0 < t < T_0 < \infty$, where $\tilde{C}_1$ and $\tilde{C}_2$ are constants.

We consider (the reduced system labelled $(1)_h$):

$$\begin{cases} u'(t) \;+\; D^{\alpha}_{0|t}(u-u_0) \;=\; |v|^q, \quad t>0 \\ \\ v'(t) \;+\; D^{\beta}_{0|t}(v-v_0) \;=\; |u|^p, \quad t>0. \end{cases}$$

No explicit solutions are available.

## 2. THE RESULT

The system $(1)_h$–(3) admits a solution $(u,v) \in \Big(C^1(0,T_{max}) \cap C([0,T_{max}))\Big)^2$ with the life span $T_{max}$, i.e., either $T_{max} = \infty$, or $T_{max} < \infty$ and $\lim\limits_{t \to T_{max}} \{|u(t)| + |v(t)|\} = +\infty$.

**THE RESULT:** Let $1 < p, q$ and $u_0 > 0$, $v_0 > 0$, then for

$(*)$ $\qquad 1 - \frac{1}{pq} \le \min\Big\{\alpha + \frac{\beta}{p}, \beta + \frac{\alpha}{q}\Big\}$

the system $(1)_h$ - (3) admits no global solutions.

**Proof.** Let $(u,v)$ be a global solution to $(1)_h$ - (3). It satisfies

$$\int_0^T |v|^q\,\varphi + u_0\left(1+\int_0^T D^{\alpha}_{t|T}\varphi\right) = -\int_0^T u\,\varphi' + \int_0^T u\,D^{\alpha}_{t|T}\varphi, \qquad (4)$$

$$\int_0^T |u|^p\,\varphi + v_0\left(1+\int_0^T D^{\beta}_{t|T}\varphi\right) = -\int_0^T v\,\varphi' + \int_0^T v\,D^{\beta}_{t|T}\varphi, \qquad (5)$$

for any non-increasing test function $\varphi \geq 0$ s.t $\varphi(0)=1$ and $\varphi(T)=0$.
The test function is chosen s.t

$$\mathcal{A} := \int_0^T \varphi^{-p'/p}\,|\varphi'|^{p'},\ \ \mathcal{B} := \int_0^T \varphi^{-q'/q}\,|\varphi'|^{q'},$$

$$\mathcal{C} := \int_0^T \varphi^{-p'/p}\,|D^{\alpha}_{t|T}\,\varphi|^{p'},\ \ \mathcal{D} := \int_0^T \varphi^{-q'/q}\,|D^{\beta}_{t|T}\,\varphi|^{q'}$$

are finite.

A part of our strategy is to estimate the left hand sides of (4) and (5) by quantities containing only the test function $\varphi$.

We write

$$\int_0^T u\,\varphi' = \int_0^T u\,\varphi^{1/p}\,\varphi^{-1/p}\,\varphi',$$

By the Hölder inequality,

$$\int_0^T u\,\varphi' \leq \left(\int_0^T |u|^p\,\varphi\right)^{1/p} \left(\int_0^T \varphi^{-p'/p}\,|\varphi'|^{p'}\right)^{1/p'},$$

and similarly

$$\int_0^T v\,\varphi' \leq \left(\int_0^T |v|^q\,\varphi\right)^{1/q} \left(\int_0^T \varphi^{-q'/q}\,|\varphi'|^{q'}\right)^{1/q'},$$

where $p + p' = pp'$ and $q + q' = qq'$.

Also, we have

$$\int_0^T u\, D^{\alpha}_{t|T}\varphi \le \left(\int_0^T |u|^p\,\varphi\right)^{\frac{1}{p}} \left(\int_0^T \varphi^{-p'/p}\,|D^{\alpha}_{t|T}\,\varphi|^{p'}\right)^{\frac{1}{p'}},$$

and

$$\int_0^T v\, D^{\beta}_{t|T}\varphi \le \left(\int_0^T |v|^q\,\varphi\right)^{\frac{1}{q}} \left(\int_0^T \varphi^{-q'/q}\,|D^{\beta}_{t|T}\,\varphi|^{q'}\right)^{\frac{1}{q'}}.$$

Now, if we set

$$\mathcal{I} = \int_0^T |u|^p\,\varphi \qquad \text{and} \qquad \mathcal{J} = \int_0^T |v|^q\,\varphi,$$

we get from (4) and (5) the estimates

$$\mathcal{J} + u_0\left(1 + \int_0^T D^{\alpha}_{t|T}\varphi\right) \le \mathcal{I}^{1/p}\left(\mathcal{A}^{1/p'} + \mathcal{C}^{1/p'}\right),$$

and

$$\mathcal{I} + v_0\left(1 + \int_0^T D^{\beta}_{t|T}\varphi\right) \le \mathcal{J}^{1/q}\left(\mathcal{B}^{1/q'} + \mathcal{D}^{1/q'}\right).$$

As $\varphi$ is decreasing and $\varphi(T) = 0$, we have

$$D^{\alpha}_{t|T}\varphi(t) = \frac{-1}{\Gamma(1-\alpha)}\int_t^T \frac{\varphi'(\sigma)}{(\sigma - t)^{\alpha}}\,d\sigma \ge 0$$

and

$$D^{\beta}_{t|T}\varphi(t) = \frac{-1}{\Gamma(1-\beta)}\int_t^T \frac{\varphi'(\sigma)}{(\sigma-t)^{\beta}}\,d\sigma \geq 0.$$

At this stage, let $\vartheta(t)$ be a non-increasing regular function such that

$$\vartheta(t) = \begin{cases} 1, & 0 \leq t \leq 1/2; \\ 0, & t \geq 1 \end{cases}$$

and choose the test function

$$\varphi(t) = \vartheta(\frac{t}{T}),$$

where $T$ is a positive real number.

We change the variable and set

$$t =: \tau T$$

so

$$\varphi'(t) = T^{-1}\varphi'(\tau),\ \ D^{\theta}_{t|T}\varphi(t) = T^{-\theta}\, D^{\theta}_{t|T}\varphi(\tau).$$

From inequalities (4) and (5), we deduce the inequalities

$$\mathcal{J} \leq \mathcal{I}^{1/p} \left(\mathcal{A}^{1/p'} + \mathcal{C}^{1/p'}\right)$$

and

$$\mathcal{I} \leq \mathcal{J}^{1/q} \left(\mathcal{B}^{1/q'} + \mathcal{D}^{1/q'}\right);$$

whereupon

$$\mathcal{J}^{1-\frac{1}{pq}} \leq \left(\mathcal{A}^{1/p'} + \mathcal{C}^{1/p'}\right) \left(\mathcal{B}^{1/q'} + \mathcal{D}^{1/q'}\right)^{1/p}$$

and

$$\mathcal{I}^{1-\frac{1}{pq}} \leq \left(\mathcal{A}^{1/p'} + \mathcal{C}^{1/p'}\right)^{1/q} \left(\mathcal{B}^{1/q'} + \mathcal{D}^{1/q'}\right).$$

Using

$$(a+b)^{1/r} \leq a^{1/r} + b^{1/r}, \quad r \geq 1,\ a>0, b>0,$$

we get the inequalities

$$\mathcal{J}^{1-\frac{1}{pq}} \leq \left(\mathcal{A}^{1/p'} + \mathcal{C}^{1/p'}\right) \left(\mathcal{B}^{1/pq'} + \mathcal{D}^{1/pq'}\right)$$

and

$$\mathcal{I}^{1-\frac{1}{pq}} \leq \left(\mathcal{A}^{1/qp'} + \mathcal{C}^{1/qp'}\right) \left(\mathcal{B}^{1/q'} + \mathcal{D}^{1/q'}\right).$$

We have

$$\begin{aligned} \mathcal{A} &\leq C_1\, T^{1-p'}, & \qquad \mathcal{B} &\leq C_2\, T^{1-q'}, \\ \mathcal{C} &\leq C_3\, T^{1-p'\alpha}, & \qquad \mathcal{D} &\leq C_4\, T^{1-q'\beta} \end{aligned}$$

for some positive constants $C_1, C_2, C_3, C_4$.

We then have the estimate (EJ)

$$\mathcal{J} \leq K_1\ (T^{s_1} + T^{s_2} + T^{s_3} + T^{s_4}) - u_0(1 + C_\alpha^* T^{1-\alpha})$$

where $K_1 = \max\{C_1, C_2, C_3, C_4\}$ and

$$\begin{aligned} s_1 &= -\left(\tfrac{1}{pq} + \tfrac{1}{p}\right), \quad & s_2 &= 1 - \tfrac{1}{p} - \alpha - \tfrac{1}{pq}, \\ s_3 &= -\left(\tfrac{1}{pq} + \tfrac{\beta}{p}\right), \quad & s_4 &= 1 - \alpha - \tfrac{1}{pq} - \tfrac{\beta}{p}. \end{aligned}$$

At this stage, we require $s_1 \leq 0, s_2 \leq 0, s_3 \leq 0, s_4 \leq 0$; this leads to

$$1 - \alpha \leq \frac{1}{pq} + \frac{\beta}{p}.$$

We have two cases:

- either $\max\{s_1, s_2, s_3, s_4\} < 0$. In this case, we let $T$ goes to infinity in (EJ), we obtain

$$0 < u_0 \leq 0.$$

This contradicts the hypotheses.

- Or, one of the $s_i$, $i = 1, 2, 3, 4$ is equal to zero; in this case, we have

$$\int_0^\infty |u(t)|^p \, dt \leq C$$

which leads to

$$\lim_{T\to\infty} \int_{T/2}^{T} |u(t)|^p \, \varphi(t) \, dt = 0.$$

Coming back to (3) and (4) and using the integral $\int_{\mathrm{supp}\varphi} |u(t)|^p \varphi$ rather than $\int_0^T |u(t)|^p \varphi$, we arrive at a contradiction too.

A similar analysis could be performed via (EI)

$\mathcal{I} \leq K_2 \, (T^{r_1} + T^{r_2} + T^{r_3} + T^{r_4}) - v_0(1 + C_\beta^* T^{1-\beta})$.

It would lead to the requirement

$$1 - \beta \leq \frac{1}{pq} + \frac{\alpha}{q}.$$

## A bound on the blow-up time

We have

$$u_0 \leq K_1 \left(T^{s_1} + T^{s_2} + T^{s_3} + T^{s_4}\right)$$

and

$$u_0 \leq \frac{K_1}{C_\alpha^*} \left(T^{s_1^*} + T^{s_1} + T^{s_3^*} + T^{s_3}\right)$$

where $s_1^* = s_1 - 1 + \alpha, \; s_3^* = s_3 - 1 + \alpha.$
These inequalities lead to

$$T_{max} \leq T_0 := \min \Bigg\{ \left(\frac{4K_1}{u_0}\right)^{s_1}, \left(\frac{4K_1}{u_0}\right)^{s_2}, \left(\frac{4K_1}{u_0}\right)^{s_3}, \left(\frac{4K_1}{u_0}\right)^{s_4}, \left(\frac{4K_1}{C_\alpha^* u_0}\right)^{s_1}, \left(\frac{4K_1}{C_\alpha^* u_0}\right)^{s_1^*}, \left(\frac{4K_1}{C_\alpha^* u_0}\right)^{s_3}, \left(\frac{4K_1}{C_\alpha^* u_0}\right)^{s_3^*} \Bigg\}.$$

and

$$T_{max} \leq T_1 := \min \Bigg\{ \left(\frac{4K_2}{u_0}\right)^{r_1}, \left(\frac{4K_2}{u_0}\right)^{r_2}, \left(\frac{4K_2}{u_0}\right)^{r_3}, \left(\frac{4K_2}{u_0}\right)^{r_4}, \left(\frac{4K_2}{C_\beta^* u_0}\right)^{r_1}, \left(\frac{4K_2}{C_\beta^* u_0}\right)^{r_1^*}, \left(\frac{4K_2}{C_\beta^* u_0}\right)^{r_3}, \left(\frac{4K_2}{C_\beta^* u_0}\right)^{r_3^*} \Bigg\}.$$

So

$$T_{max} \leq \min \{T_0, T_1\}.$$

**Remark:** The analysis may be efficient for more general systems

$$au'(t)+D^{\alpha}_{0|t}(u^m-u^m_0)+bD^{\delta}_{0|t}(v^l-v^l_0)=f_1(t,|u|,|v|)$$

$$cv'(t)+D^{\beta}_{0|t}(v^n-v^n_0)+dD^{\sigma}_{0|t}(u^k-u^k_0)=f_2(t,|u|,|v|)$$

with either

$$\begin{cases} f_1(t,|u|,|v|)\geq c_1t^{\vartheta_1}|u|^{p_1}+c_2t^{\vartheta_2}|v|^{p_2}, \\ \\ f_2(t,|u|,|v|)\geq c_3t^{\vartheta_3}|u|^{p_3}+c_4t^{\vartheta_4}|v|^{p_4}, \end{cases}$$

or

$$\begin{cases} f_1(t,|u|,|v|)\geq d_1t^{\vartheta_1}|u|^{p_1}\,|v|^{p_2}, \\ \\ f_2(t,|u|,|v|)\geq d_2t^{\vartheta_2}|u|^{p_3}\,|v|^{p_4}. \end{cases}$$

with $0<d_1,d_2,\vartheta_1,\vartheta_2, 1<p_1,p_2,p_3,p_4$.