



Commande - Robotique
Ordres Non Entiers



UMR 5131

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H_∞ -Norm Computation of Fractional Systems

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Outline

Part 1: LMI tools for Stability Analysis of Fractional Systems

- 1 – LMI and integer order systems stability
- 2 – Stability of fractional order systems
- 3 – First method
- 4 – Second method
- 5 – Third method

Part 2: H_∞ -norm computation of fractional systems

- 1 – H_∞ -Norm: definition
- 2 – Extension of Bounded Real Lemma
- 3 – Computation based on a derived fractional system stability analysis

1 – LMI and integer order systems stability

1 – Presentation of LMI

2 – Properties of LMI

3 – Lyapunov's second method: analysis of integer systems stability

1- LMI and integer order systems stability

2- Stability of fractional systems

3- First method

4- Second method

5- Third method

Linear Matrix Inequalities (LMI): presentation

- LMI: $P(x) = P_0 + x_1 P_1 + \dots + x_r P_r > 0$

P_i are symmetric matrices

- Stability analysis of integer systems (Lyapunov), Systems norms calculation (bounded real lemma)

- Graphic tools, algebraic methods, **numeric tools**

Convex optimization: interior point methods

➔ Global optimum

1- LMI and integer order systems stability

2- Stability of fractional systems

3- First method

4- Second method

5- Third method

Properties of LMI

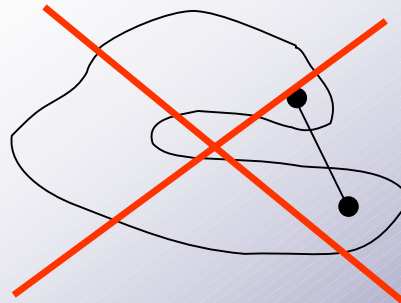
- Several LMIs can be gathered into one LMI:

$$A(x) > 0 \text{ and } B(x) < 0 \Leftrightarrow \begin{pmatrix} A(x) & 0 \\ 0 & -B(x) \end{pmatrix} > 0$$

- LMI with complex coefficients can be represented by a LMI with real ones:

$$P(x) = S(x) + jT(x) > 0 \Leftrightarrow \begin{pmatrix} S(x) & T(x) \\ -S(x) & S(x) \end{pmatrix} > 0$$

- For a LMI to be solved, the domain must be convex



1- LMI and integer order systems stability

2- Stability of fractional systems

3- First method

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Lyapunov's second method: stability analysis of integer systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \xrightarrow{\text{Autonomous system}} \begin{cases} \dot{x}(t) = Ax(t) \\ x(t_0) = x_0 \end{cases}$$

Asymptotic stability if $\exists V(x) > 0 : \dot{V}(x) < 0$

$$V(x) = x^T(t)Px(t) \quad \rightarrow \quad \dot{V}(x) = \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t)$$

$$\dot{V}(x) = (Ax(t))^T Px(t) + x^T(t)P(Ax(t))$$

$$\dot{V}(x) = x(t)^T (A^T P + PA)x(t)$$

Asymptotic stability if

$$\boxed{\exists P > 0 : A^T P + PA < 0}$$



LMI Feasibility problem

1- LMI and integer order systems stability

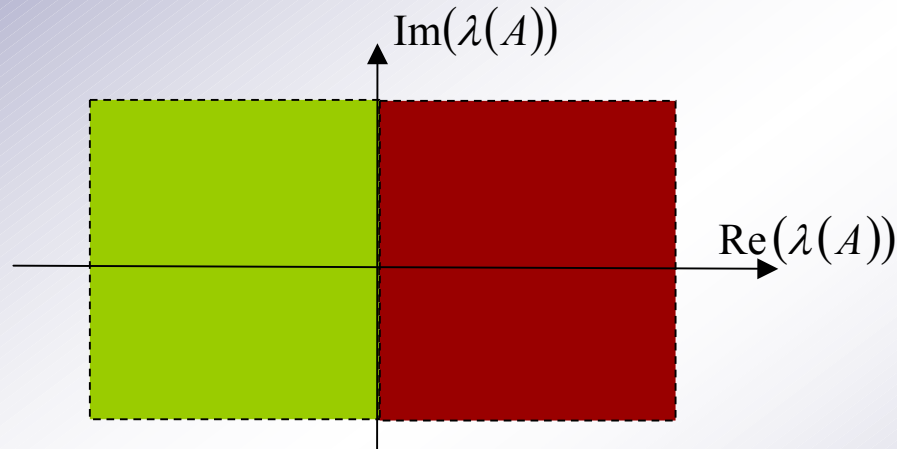
2- Stability of fractional systems

3- First method

4- Second method

5- Third method

Lyapunov's second method: stability analysis of integer systems



Equivalence:

- Integer order system with state matrix A is asymptotically stable
- The eigenvalues of matrix A lie in the left half complex plane
- $A < 0$
- $\exists P > 0 : A^T P + P A < 0$

1- LMI and integer order systems stability

2- Stability of fractional systems

3- First method

4- Second method

5- Third method

Summary

- Efficient tools are available to solve LMI
- The domain must be convex
- Using Lyapunov's method, stability of integer systems can be verified using LMI
- The left half complex plane can be described by a LMI feasibility problem

2 – Stability of fractional order systems

1 – Matignon's theorem

2 – Stability domain

- 1- LMI and integer order systems stability
- 2- Stability of fractional systems
- 3- First method

- 4- Second method
- 5- Third method

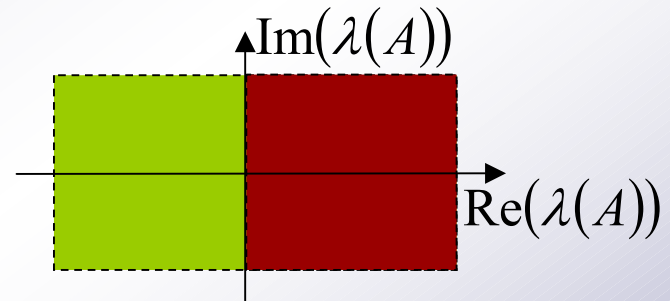
Matignon's theorem

$$(S): \begin{cases} x^{(n)}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

(S) is asymptotically stable

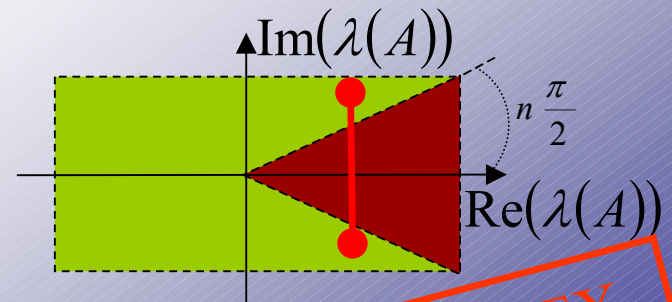
$$\Leftrightarrow \forall i, |\arg(\lambda_i(A))| > n \frac{\pi}{2}$$

- $n = 1$



■ Stability domain

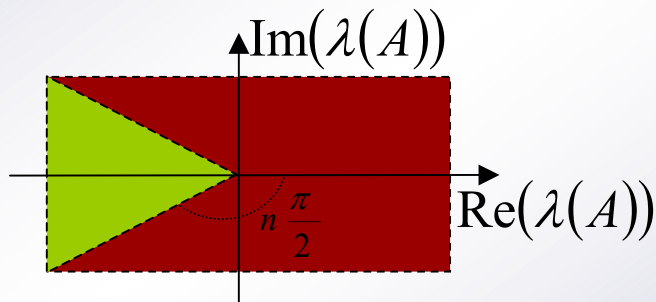
- $n < 1$



■ Stability domain

NOT CONVEX

- $n > 1$



■ Stability domain

- 1- LMI and integer order systems stability
- 2- **Stability of fractional systems**
- 3- First method

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Stability domain / Equivalent integer order system

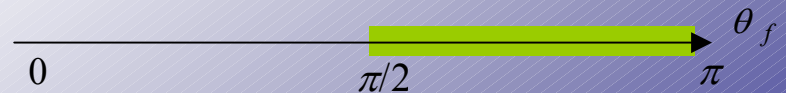
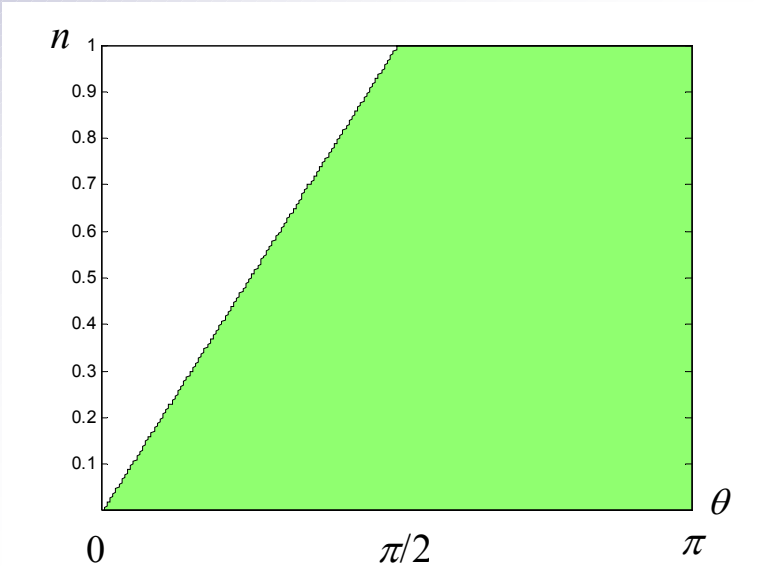
Fractional system:

State matrix: A
 $\arg(\lambda(A)) = \theta$



Equivalent integer order system:

State matrix: A_f
 $\arg(\lambda(A_f)) = \theta_f$



1- LMI and integer order systems stability

2- **Stability of fractional systems**

3- First method

4- Second method

5- Third method

Summary

- Matignon's theorem gives a criteria to verify stability of fractional systems
- The theorem is a generalization of criteria for stability of integer systems
- For $n > 1$, LMI tools have already been developed and are available in literature
- For $n < 1$, the domain is not convex and LMI can not be directly derived for stability analysis of fractional systems: an equivalent integer order system is to be found

3 – First method

1 – Equivalent integer order system

2 – LMI condition 1

3 – Analysis of the method

1- LMI and integer order systems stability

2- Stability of fractional systems

3- **First method**

4- Second method

5- Third method

Equivalent integer order system

$$\begin{cases} x^{(n)}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \xrightarrow{\text{Autonomous system}} \begin{cases} x^{(n)}(t) = Ax(t) \\ x(t_0) = x_0 \end{cases}$$

Laplace transform: $s^n X(s) = AX(s) + I^{1-n} x_0$

- $\frac{1}{n} = q \in N$ $s^{2n} X(s) = s^n s^n X(s) = A s^n X(s) + s^n I^{1-n} x_0$

$$s^{2n} X(s) = A^2 X(s) + A I^{1-n} x_0 + s^n I^{1-n} x_0$$

$$\rightarrow sX(s) = A^q X(s) + \sum_{i=1}^q A^{i-1} s^{1-in} I^{1-n} x_0$$

- $\frac{m}{n} = q \in N, m \in N$ $\rightarrow s^m X(s) = A^q X(s) + \sum_{i=1}^q A^{i-1} s^{m-in} I^{1-n} x_0$

1- LMI and integer order systems stability

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Stability criteria

state space representation:

$$\begin{cases} \dot{z}(t) = A_f z(t) + B_f \Delta(t) \\ x(t) = C_f z(t) \end{cases}$$

$$A_f = \begin{pmatrix} 0 & \dots & 0 & A^{\frac{1}{n}} \\ A^{\frac{1}{n}} & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & A^{\frac{1}{n}} & 0 \end{pmatrix}$$

$$B_f = \begin{pmatrix} 1 & A^{\frac{1}{m}} & \dots & A^{\frac{q-1}{m}} \\ & & 0 & \end{pmatrix}$$

$$C_f = (0 \quad \dots \quad 0 \quad 1)$$

$$\Delta(t) = I^{1-n} x_0 \begin{pmatrix} \delta^{m-n} \\ \delta^{m-2n} \\ \vdots \\ \delta \end{pmatrix}$$

➔ Integer system: stability is determined through

$$\forall i, \operatorname{Re}(\lambda_i(A_f)) < 0$$

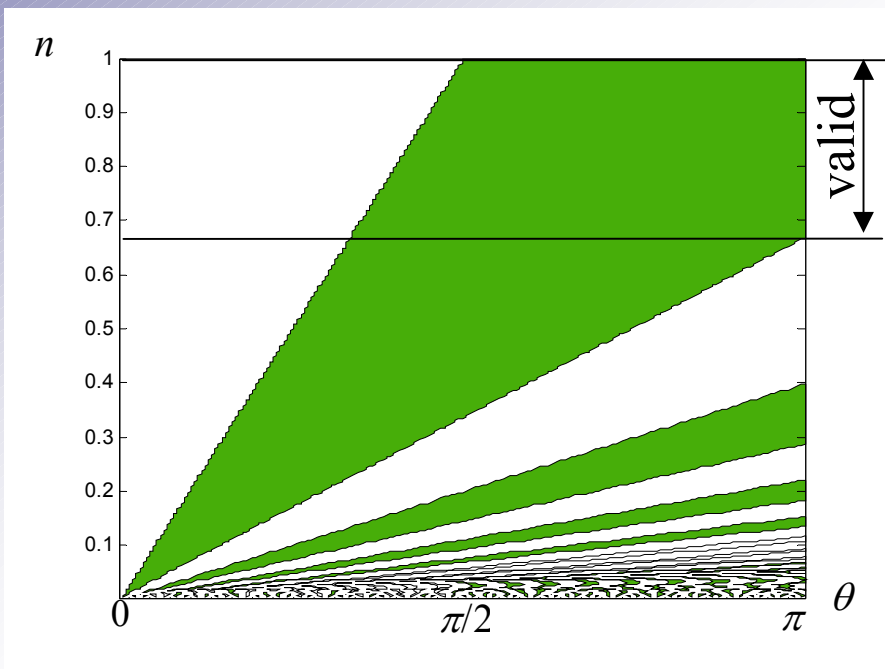
$$\forall i, \operatorname{Re}\left(\lambda_i\left(A^{\frac{1}{n}}\right)\right) < 0 \Leftrightarrow \exists P > 0 : \begin{pmatrix} A^{\frac{1}{n}} \\ A^{\frac{1}{n}} \end{pmatrix}^T P + P \begin{pmatrix} A^{\frac{1}{n}} \\ A^{\frac{1}{n}} \end{pmatrix} < 0 \rightarrow \text{LMI Feasibility problem}$$

- 1- LMI and integer order systems stability
- 2- Stability of fractional systems
- 3- **First method**

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Validity of the method 1/2

Pole localization of: $G(s) = \frac{1}{(1 + e^{i\theta} s^n)(1 + e^{-i\theta} s^n)}, 0 \leq \theta \leq \pi$



Stability domain

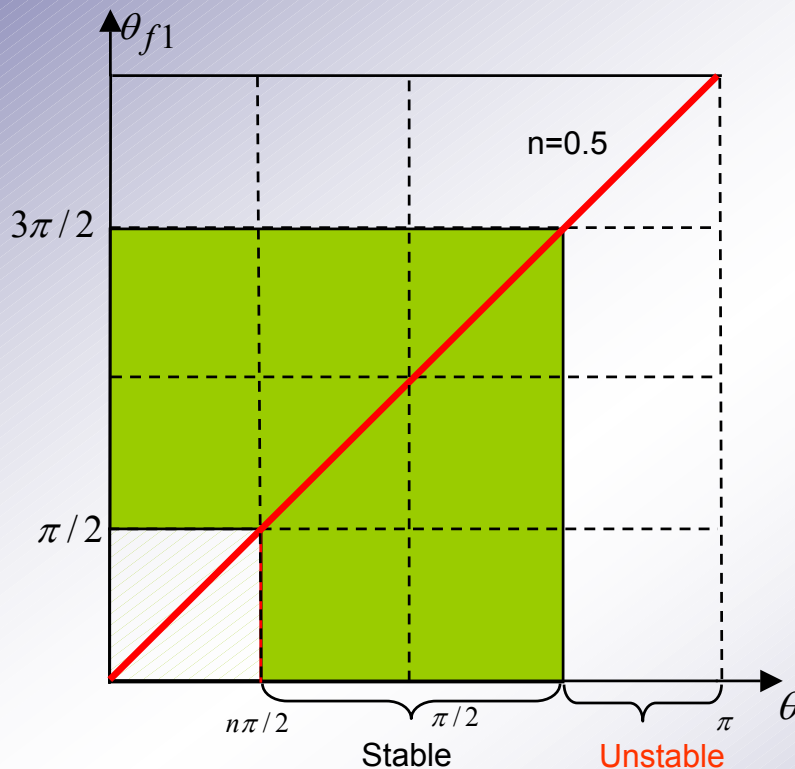
Sufficient but not necessary condition

- 1- LMI and integer order systems stability
- 2- Stability of fractional systems
- 3- **First method**

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- 5- Third method

Validity of the method 2/2

function $f_1: \theta \xrightarrow{f_1} \theta_{f1} = \frac{\theta}{n}$



The entire stability domain is not identified



Find a function that associates:

- Fractional stability domain to a subset of the integer stability domain
- And Fractional instability domain to a subset of the integer instability domain



1- LMI and integer order systems stability

2- Stability of fractional systems

3- First method

4- Second method

5- Third method

Summary

- Method 1 analyses stability of an integer system derived from a fractional one
- The method leads to a sufficient but not necessary condition
- A new function between integer and fractional stability domains must be defined

4 – Second method

1 – Valid function

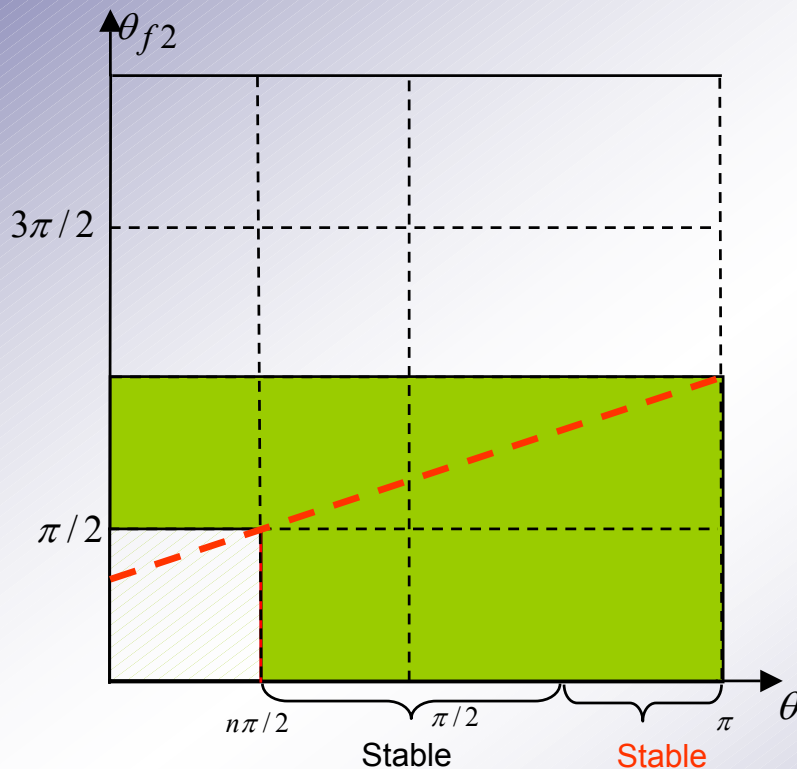
2 – LMI condition 2

- 1- LMI and integer order systems stability
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Valid function

The new function $f_2: \theta \xrightarrow{f_2} \theta_{f2} = \frac{\theta}{2-n} + \frac{1-n}{2-n} \pi$



■ Identified stability domain

let $a = e^{i\theta}$ and $b = e^{i\theta_2}$

$\theta = -i \ln a$ $b = e^{i\left(\frac{\theta}{2-n} + \frac{1-n}{2-n} \pi\right)}$

$$\ln b = i \left(\frac{-i \ln a}{2-n} + \frac{1-n}{2-n} \pi \right)$$

$$\ln b = \frac{\ln a}{2-n} + \frac{1-n}{2-n} i\pi$$

$$\ln b = \frac{\ln a}{2-n} + \frac{2-n-1}{2-n} i\pi$$

$$\ln b = \ln a^{\frac{1}{2-n}} - \frac{i\pi}{2-n} + i\pi$$

$$b = a^{\frac{1}{2-n}} e^{\frac{i\pi}{2-n}} e^{i\pi}$$



$$b = -(-a)^{\frac{1}{2-n}}$$

- 1- LMI and integer order systems stability
- 2- Stability of fractional systems
- 3- First method

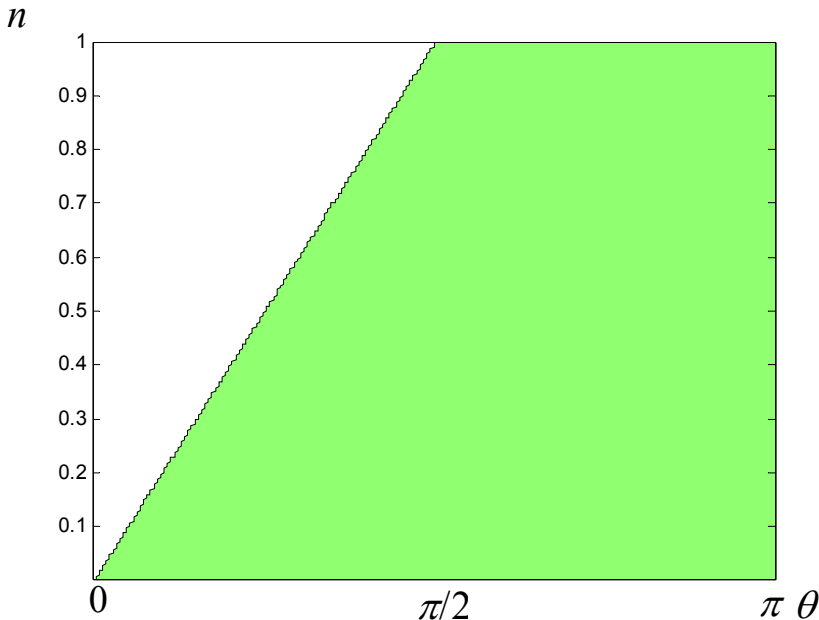
- 4- Second method
- 5- Third method

LMI condition 2

Lyapunov's method on integer system with state matrix A_{f2} :

$$A_{f2} = -(-A)^{\frac{1}{2-n}}$$

$$\exists P > 0 : \left(-(-A)^{\frac{1}{2-n}} \right)^T P + P \left(-(-A)^{\frac{1}{2-n}} \right) < 0 \quad \rightarrow \quad \text{LMI Feasibility problem}$$



Pole localization of:

$$G(s) = \frac{1}{(1 + e^{i\theta} s^n)(1 + e^{-i\theta} s^n)}, 0 \leq \theta \leq \pi$$

Sufficient and necessary condition

■ Identified stability domain

- 1- LMI and integer order systems stability
- 2- Stability of fractional systems
- 3- First method

- 4- **Second method**
- 5- Third method

Summary

- Method 2 analyses stability of an integer system derived from a fractional one
- The method leads to a sufficient AND necessary condition
- The function is non linear in relation to state matrix A

5 – Third method

1 – Duality

2 – LMI condition 3

- 1- LMI and integer order systems stability
- 2- Stability of fractional systems
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- 5- **Third method**

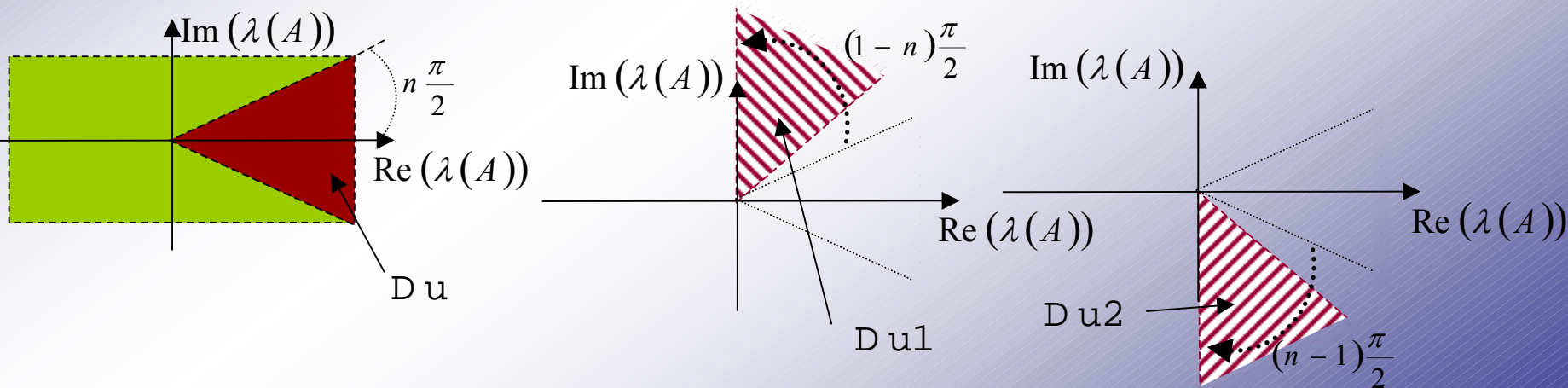
Duality / unstability domain

Instead of trying to identify the stable domain, one can characterize the instable one (convex) D_u

Fractional system is stable iff

$$\nexists \lambda \in D_u, q \in C^m : (\lambda I - A)q = 0, q \neq 0$$

- Geometric characterization of D_u



$$\lambda \in D_u \text{ iff } \text{Re}(\lambda e^{i(1-n)\pi}) \geq 0 \text{ and } \text{Re}(\lambda e^{i(n-1)\pi}) \geq 0$$

- 1- LMI and integer order systems stability
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- Characterization of D_u using inequalities

$$\forall \alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0 \Leftrightarrow \frac{\alpha + \alpha^*}{2} > 0 \Leftrightarrow \alpha + \alpha^* > 0$$

$$\lambda \in D_u \text{ iff } \operatorname{Re}(\lambda e^{i(1-n)\pi}) \geq 0 \text{ and } \operatorname{Re}(\lambda e^{i(n-1)\pi}) \geq 0$$

$$(e^{i(1-n)\pi})^* = e^{i(n-1)\pi} = r^*$$

$$\Rightarrow \lambda \in D_u \text{ iff } \begin{cases} \lambda r + \lambda^* r^* \geq 0 \\ \lambda r^* + \lambda^* r \geq 0 \end{cases}$$

- Associated stability criteria

$$\text{Fractional system is stable iff } \exists \lambda \in \mathbb{C}, q \in \mathbb{C}^m, \begin{cases} \lambda r + \lambda^* r^* \geq 0 \\ \lambda r^* + \lambda^* r \geq 0 \end{cases} : (\lambda I - A)q = 0, q \neq 0$$

Complex poles are conjugate

$$\text{Fractional system is stable iff } \exists \lambda \in \mathbb{C}, q \in \mathbb{C}^m, \lambda r + \lambda^* r^* \geq 0 : (\lambda I - A)q = 0, q \neq 0$$

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LMI condition 3

- Characterization using LMI

Lemma: There exists a vector $p = \alpha q \neq 0$ for some $\alpha + \alpha^* \geq 0$ if and only if $pq^* + qp^* \geq 0$

\swarrow
 $\alpha = \lambda r$

Fractional system is stable iff $\exists p = \lambda r q \neq 0, q \in \mathbb{C}^m, pq^* + qp^* \geq 0 : (\lambda I - A)q = 0, q \neq 0$

\swarrow \nwarrow
 $p = \lambda r q$

Fractional system is stable iff $\exists q \neq 0, q \in \mathbb{C}^m, \lambda r q q^* + q q^* A^T r^* \geq 0 : (\lambda I - A)q = 0, q \neq 0$

\swarrow \nwarrow \circ
 $q q^* = Q$ $\lambda q = A q$

$\exists Q \neq 0, \text{rank}(Q) = 1 : r A Q + Q A^T r^* \geq 0$



LMI Feasibility problem

6 – Conclusion

- Three methods:
 - Method 1: based on the derivation of an integer system
 - Method 2: based on geometric analysis
 - Method 3: based on a dual approach

Sufficient but not necessary

Function is not linear

Outline

Part 1: LMI tools for Stability Analysis of Fractional Systems

- 1 – LMI and integer order systems stability
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Part 2: H_∞ -norm computation of fractional systems

- 1 – H_∞ -Norm: definition
- 2 – Extension of Bounded Real Lemma
- 3 – Computation based on a derived fractional system stability analysis

1 – H_∞ -Norm: definition

1 – Definition

2 – Motivations

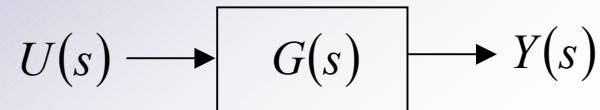
1- H_2 -Norm: Definition

2- Extension of Bounded Real Lemma

3- Computation based on a derived fractional system stability analysis

Definition

Consider a transfer function $G(s)$:



- First definition:

H_∞ -norm is the induced L_2 -norm: $\|G(s)\|_\infty = \sup_{U(s) \in H_2} \frac{\|Y(s)\|_2}{\|U(s)\|_2}$

where $\|U(s)\|_2 = \left(\int_0^{+\infty} \|u(t)\|^2 dt \right)^{1/2}$ or (Parseval) $\|U(s)\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \|U(j\omega)\|^2 d\omega \right)^{1/2}$

- Second definition:

$$\|G(s)\|_\infty = \sup_{\omega \in R} \sigma(G(j\omega))$$

where $\sigma(G(j\omega)) = \left(\text{spec}(G(j\omega)^* G(j\omega)) \right)^{1/2}$

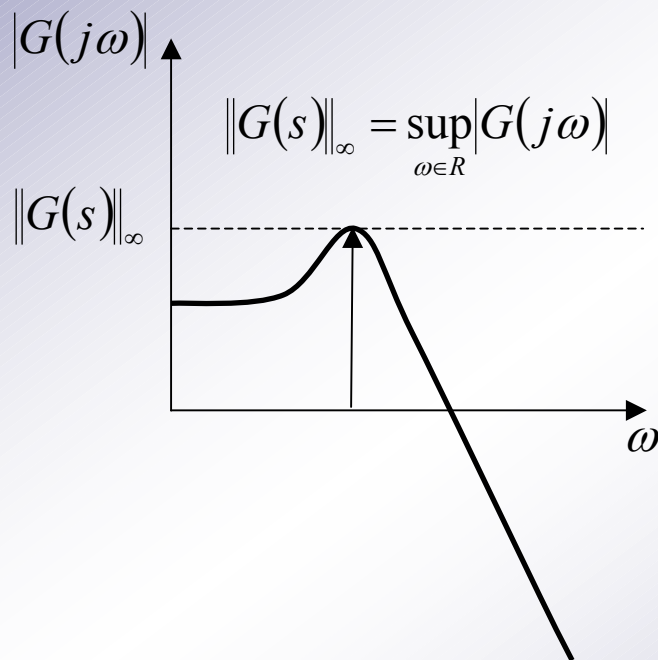
1- H_∞ -Norm: Definition

2- Extension of Bounded Real Lemma

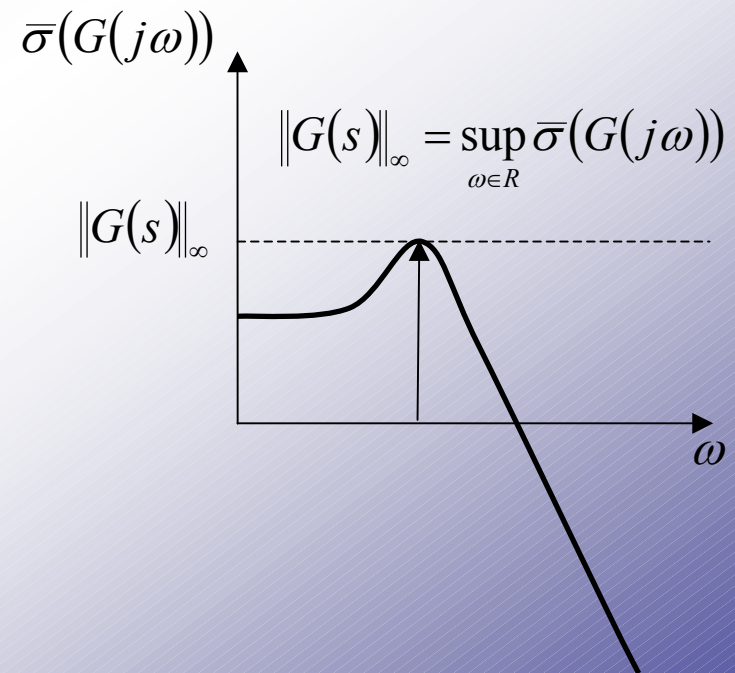
3- Computation based on a derived fractional system stability analysis

Definition

SISO Case:



MIMO Case:



1- H_{∞} -Norm: Definition

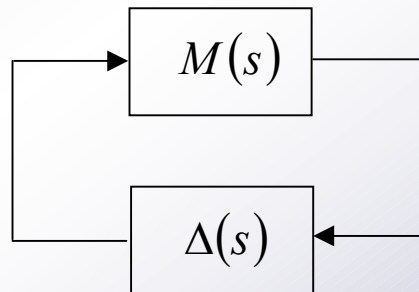
2- Extension of Bounded Real Lemma

3- Computation based on a derived fractional system stability analysis

Motivation

- Study of LPV/LTV systems as frequency analysis is not available
- Small gain theorem

If $M(s)$ and $\Delta(s)$ are stable, the system is stable for every $\Delta(s)$ such that $\|\Delta(s)\|_{\infty} < \alpha$ if $\|M(s)\|_{\infty} \leq \alpha^{-1}$.



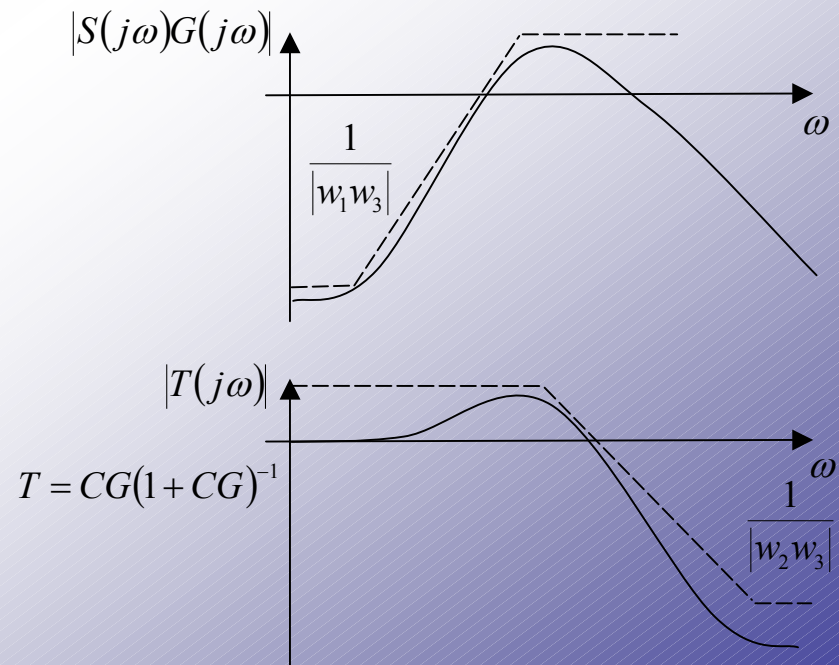
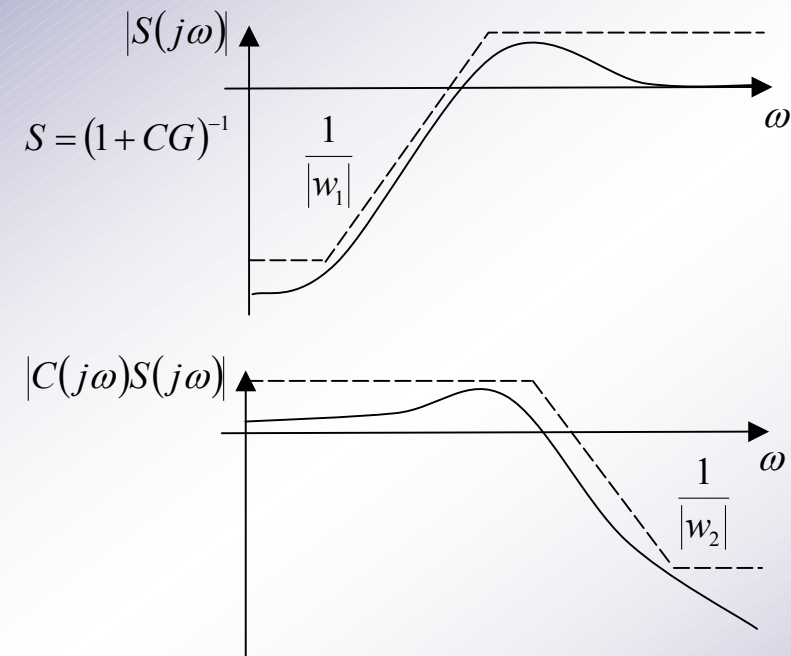
1- H_∞ -Norm: Definition

2- Extension of Bounded Real Lemma

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Motivation

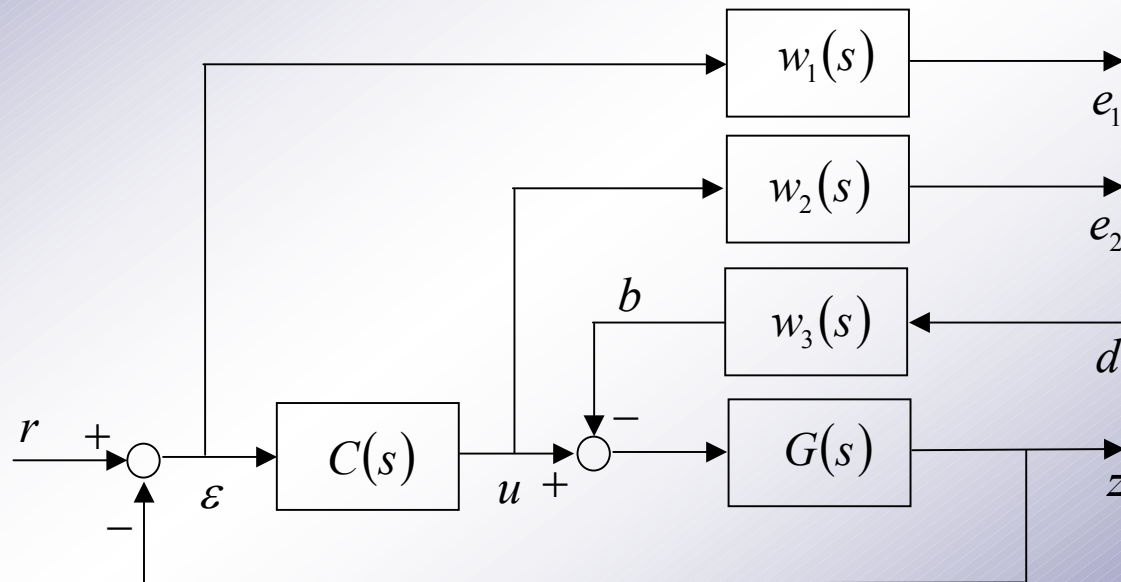
- Study of LPV/LTV systems as frequency analysis is not available
- Small gain theorem
- Constraints on sensitivity functions



- 1- H_∞ -Norm: Definition
- 2- Extension of Bounded Real Lemma
- 3- Computation based on a derived fractional system stability analysis

Motivation

- Study of LPV/LTV systems as frequency analysis is not available
- Small gain theorem
- Constraints on sensitivity functions



$$\left\| \begin{pmatrix} w_1 S & w_1 S G w_3 \\ w_2 C S & w_2 T w_3 \end{pmatrix} \right\|_\infty < 1$$

1- H_2 -Norm: Definition

2- Extension of Bounded Real Lemma

3- Computation based on a derived fractional system stability analysis

Summary

- Two equivalent formulations exist
- H_∞ norm is efficient to deal with LPV/LTV systems
- H_∞ norm enables formulation of various problems in control
- Exact computation of H_∞ norm does not exist

2 – Extension of Bounded Real Lemma

1 – Bounded Real Lemma

2 – Extended Bounded Real Lemma

3 – Particular case: SISO systems

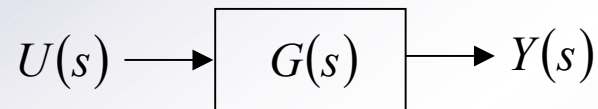
1- H_2 -Norm: Definition

2- **Extension of Bounded Real Lemma**

3- Computation based on a derived fractional system stability analysis

Bounded Real Lemma

Consider a transfer function $G(s)$:



- First definition:

$$H_2\text{-norm is the induced } L_2\text{-norm: } \|G(s)\|_\infty = \sup_{U(s) \in H_2} \frac{\|Y(s)\|_2}{\|U(s)\|_2}$$

$$\text{where } \|U(s)\|_2 = \left(\int_0^{+\infty} \|u(t)\|^2 dt \right)^{1/2} \text{ or (Parseval) } \|U(s)\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \|U(j\omega)\|^2 d\omega \right)^{1/2}$$

- Second definition:

$$\|G(s)\|_\infty = \sup_{\omega \in R} \bar{\sigma}(G(j\omega))$$

$$\text{where } \sigma(G(j\omega)) = \left(\text{spec}(G(j\omega)^* G(j\omega)) \right)^{1/2}$$

1- H_2 -Norm: Definition

2- **Extension of Bounded Real Lemma**

3- Computation based on a derived fractional system stability analysis

Bounded Real Lemma

Bounded Real Lemma (integer order systems):

Integer order system given by:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} ,$$

is stable and its H_∞ -norm is bounded by γ if and only if there exists a symmetric positive definite matrix P such that:

$$\begin{pmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} < 0 .$$

That is,

$$\forall T \geq 0, \int_0^T y(t)^T y(t) dt \leq \gamma^2 \int_0^T u(t)^T u(t) dt$$

1- H_∞ -Norm: Definition,

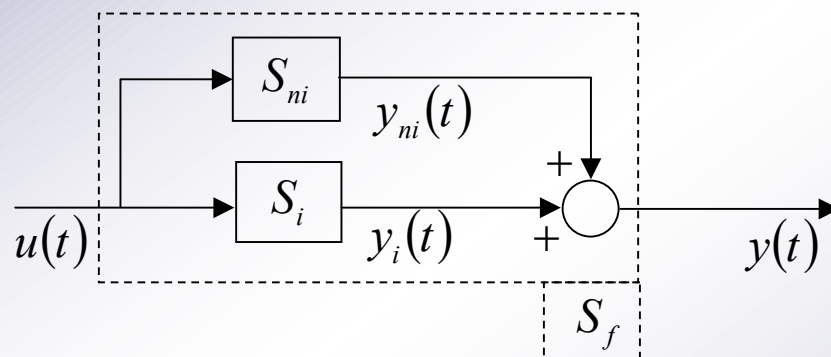
2- **Extension of Bounded Real Lemma**

3- Computation based on a derived fractional system stability analysis

Extended Bounded Real Lemma

Extension to fractional systems

Consider



And suppose $\|S_{ni}\|_\infty \leq r_2$, known.

From $\forall T \geq 0, \int_0^T y(t)^T y(t) dt \leq \gamma^2 \int_0^T u(t)^T u(t) dt$ and as $y(t) = y_{ni}(t) + y_i(t)$

$$\forall T \geq 0, \int_0^T (y_i(t) + y_{ni}(t))^T (y_i(t) + y_{ni}(t)) dt \leq \gamma^2 \int_0^T u(t)^T u(t) dt$$

- 1- H_∞ -Norm: Definition
- 2- **Extension of Bounded Real Lemma**
- 3- Computation based on a derived fractional system stability analysis

Extended Bounded Real Lemma

$$(y_i + y_{ni})^T (y_i + y_{ni}) = y_i^T y_i + y_{ni}^T y_{ni} + 2y_{ni}^T y_i$$

$$\text{but } (y_i - y_{ni})^T (y_i - y_{ni}) = y_i^T y_i + y_{ni}^T y_{ni} - 2y_{ni}^T y_i$$

$$\text{and } (y_i - y_{ni})^T (y_i - y_{ni}) \geq 0$$

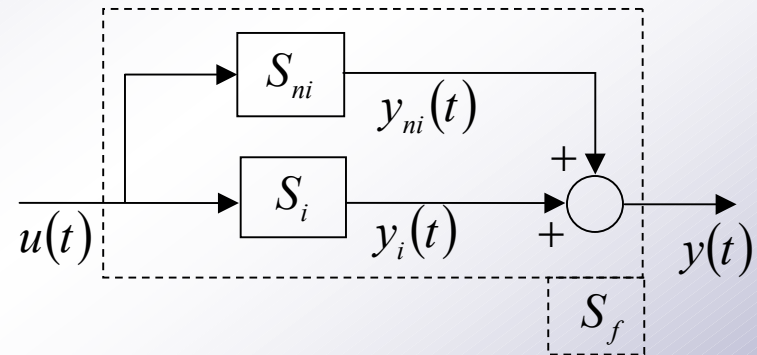
$$\text{gives } y_i^T y_i + y_{ni}^T y_{ni} \geq 2y_{ni}^T y_i$$

$$\text{Thus } y_i^T y_i + y_{ni}^T y_{ni} + 2y_{ni}^T y_i \leq 2y_i^T y_i + 2y_{ni}^T y_{ni}$$

$$(y_i + y_{ni})^T (y_i + y_{ni}) \leq 2y_i^T y_i + 2y_{ni}^T y_{ni}$$

$$\forall T \geq 0, \int_0^T 2y_i(t)^T y_i(t) dt + 2 \int_0^T y_{ni}(t)^T y_{ni}(t) dt \leq \gamma^2 \int_0^T u(t)^T u(t) dt$$

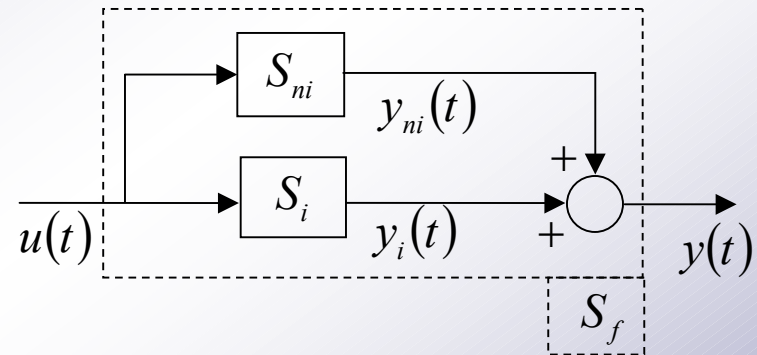
$$\Rightarrow \forall T \geq 0, \int_0^T y(t)^T y(t) dt \leq \gamma^2 \int_0^T u(t)^T u(t) dt$$



- 1- H_2 -Norm: Definition
- 2- **Extension of Bounded Real Lemma**
- 3- Computation based on a derived fractional system stability analysis

Extended Bounded Real Lemma

$$\text{As } \|S_{ni}\|_\infty \leq r_2, \forall T \geq 0, \int_0^T y_{ni}(t)^T y_{ni}(t) dt \leq r_2^2 \int_0^T u(t)^T u(t) dt$$



$$\forall T \geq 0, 2 \int_0^T y_i(t)^T y_i(t) dt + 2r_2^2 \int_0^T u(t)^T u(t) dt \leq \gamma^2 \int_0^T u(t)^T u(t) dt$$

$$\Rightarrow \forall T \geq 0, \int_0^T y(t)^T y(t) dt \leq \gamma^2 \int_0^T u(t)^T u(t) dt$$

$$\forall T \geq 0, 2 \int_0^T y_i(t)^T y_i(t) dt \leq (\gamma^2 - 2r_2^2) \int_0^T u(t)^T u(t) dt$$

$$\Rightarrow \forall T \geq 0, \int_0^T y(t)^T y(t) dt \leq \gamma^2 \int_0^T u(t)^T u(t) dt$$

- 1- H_∞ -Norm: Definition
- 2- **Extension of Bounded Real Lemma**
- 3- Computation based on a derived fractional system stability analysis

Extended Bounded Real Lemma

Bounded Real Lemma

Integer order system given by:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

is stable and its H_∞ -norm is bounded by γ if and only if there exists a symmetric positive definite matrix P such that:

$$\begin{pmatrix} A^T P + PA & PB & C^T \\ B^T P & -\mathcal{I} & D^T \\ C & D & -\mathcal{I} \end{pmatrix} < 0.$$

Extended Bounded Real Lemma

Fractional order system given by

$$S_f = S_i + S_{ni},$$

where $\|S_{ni}\|_\infty < r_2$

and S_i is given by:

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u \\ y_i &= C_i x_i + D_i u \end{aligned}$$

is stable and its H_∞ -norm is bounded by γ if and only if there exists a symmetric positive definite matrix P such that:

$$\begin{pmatrix} A_i^T P + PA_i & PB_i & C_i^T \\ B_i^T P & -(\gamma^2 - 2r_2^2)^{1/2} I & D_i^T \\ C_i & D_i & -(\gamma^2 - 2r_2^2)^{1/2} I \end{pmatrix} < 0.$$

- 1- H_∞ -Norm: Definition
- 2- **Extension of Bounded Real Lemma**
- 3- Computation based on a derived fractional system stability analysis

Particular case: SISO systems

Let S be a SISO system whose transfer function is $G(s)$. Then

$$\|G(s)\|_\infty < \|R\|_{L_1} \leq r_2$$

where R is the impulse response of S .

A decomposition enabling analytic calculation of L_1 -gain of the non-integer subsystem is available in

“*Contributions à la théorie structurelle et la commande des systèmes linéaires fractionnaires*”,

R. Hotzel, PhD Thesis, 1998,
Université de Paris Sud, France

Extended Bounded Real Lemma

Fractional order system given by

$$S_f = S_i + S_{ni} ,$$

where $\|S_{ni}\|_\infty < r_2$

and S_i is given by:

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u \\ y_i &= C_i x_i + D_i u \end{aligned} ,$$

is stable and its H_∞ -norm is bounded by γ if and only if there exists a symmetric positive definite matrix P such that:

$$\begin{pmatrix} A_i^T P + P A_i & P B_i & C_i^T \\ B_i^T P & -(\gamma^2 - 2r_2^2)^{1/2} I & D_i^T \\ C_i & D_i & -(\gamma^2 - 2r_2^2)^{1/2} I \end{pmatrix} < 0 .$$

1- H_2 -Norm: Definition

2- **Extension of Bounded Real Lemma**

3- Computation based on a derived fractional system stability analysis

Summary

- The method necessitates a bound on the fractional subsystem H_∞ norm
- Analytic formulation of this bound is available in literature
- H_2 norm is expressed through LMI constraint

3 – Computation based on a derived fractional system stability analysis

1 – Stability condition

2 – Derived fractional system computation

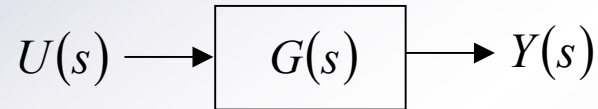
1- H_2 -Norm: Definition

2- Extension of Bounded Real Lemma

3- **Computation based on a derived fractional system stability analysis**

Stability condition

Consider a transfer function $G(s)$:



- First definition:

H_∞ -norm is the induced L_2 -norm: $\|G(s)\|_\infty = \sup_{U(s) \in H_2} \frac{\|Y(s)\|_2}{\|U(s)\|_2}$

where $\|U(s)\|_2 = \left(\int_0^{+\infty} \|u(t)\|^2 dt \right)^{1/2}$ or (Parseval) $\|U(s)\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \|U(j\omega)\|^2 d\omega \right)^{1/2}$

- Second definition:

$$\|G(s)\|_\infty = \sup_{\omega \in R} \bar{\sigma}(G(j\omega))$$

where $\sigma(G(j\omega)) = \left(\text{spec}(G(j\omega)^* G(j\omega)) \right)^{1/2}$

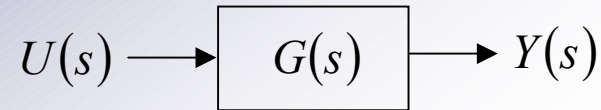
1- H_∞ -Norm: Definition

2- Extension of Bounded Real Lemma

3- **Computation based on a derived fractional system stability analysis**

Stability condition

Consider fractional system S whose transfer matrix is $G(s)$



Given commensurate order hypothesis:

$$(S): \begin{cases} D^\nu x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \quad \text{and} \quad G(s) = C(s^\nu I - A)^{-1} B + D$$

From definition 2:

$$\|G(s)\|_\infty < \gamma \quad \text{iff} \quad \sup_{\omega \in R} \bar{\sigma}(G(j\omega)) \leq \gamma$$

$$\text{iff} \quad \forall \omega \in R, \forall i, \quad \sigma_i(G(j\omega)) \leq \gamma$$

$$\text{but} \quad \sigma(G(j\omega)) = \left(\text{spec}(G(j\omega)^* G(j\omega)) \right)^{1/2}$$

$$\text{thus} \quad \|G(s)\|_\infty < \gamma \quad \text{iff} \quad \forall \omega \in R, \forall i, \quad \lambda_i(G(-j\omega)^T G(j\omega)) \leq \gamma^2$$

1- H_2 -Norm: Definition

2- Extension of Bounded Real Lemma

3- **Computation based on a derived fractional system stability analysis**

Stability condition

$$\begin{aligned} \text{thus } \|G(s)\|_\infty < \gamma \text{ iff } \forall \omega \in R, \forall i, \lambda_i(G(-j\omega)^T G(j\omega)) \leq \gamma^2 \\ \text{iff } \forall \omega \in R, \forall i, \lambda_i(\gamma^2 I - G(-j\omega)^T G(j\omega)) > 0 \end{aligned}$$

↓ LMI

$$\|G(s)\|_\infty < \gamma \text{ iff } \forall \omega \in R, \gamma^2 I - G(-j\omega)^T G(j\omega) > 0$$

As $\forall \omega \in R, \gamma > \bar{\sigma}(D)$ and as $\lim_{\omega \rightarrow +\infty} \gamma^2 I - G(-j\omega)^T G(j\omega) = \gamma^2 I - D^T D > 0$

$\|G(s)\|_\infty < \gamma$ iff $\gamma^2 I - G(-s)^T G(s)$ has no zero on the imaginary axis

$\|G(s)\|_\infty < \gamma$ iff $(\gamma^2 I - G(-s)^T G(s))^{-1}$ is asymptotically stable

- 1- H_∞ -Norm: Definition
- 2- Extension of Bounded Real Lemma
- 3- **Computation based on a derived fractional system stability analysis**

Derived fractional system computation

$$\|G(s)\|_\infty < \gamma \quad \text{iff} \quad (\gamma^2 I - G(-s)^T G(s))^{-1} \text{ is asymptotically stable}$$

As $G(s) = C(s^\nu I - A)^{-1} B + D$

$$G(-s) = C((-s)^\nu I - A)^{-1} B + D$$

$$\Downarrow \quad (-s)^\nu = (e^{j\pi} s)^\nu = e^{j\nu\pi} s^\nu$$

$$G(-s) = C(e^{j\nu\pi} s^\nu I - A)^{-1} B + D$$

$$G(-s) = C(s^\nu I - e^{-j\nu\pi} A)^{-1} e^{-j\nu\pi} B + D$$

$$\gamma^2 I - G(-s)^T G(s) : \begin{cases} D^\nu x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$$A' = \begin{pmatrix} A & 0 \\ C^T C & e^{-j\nu\pi} \end{pmatrix}$$

$$B' = \begin{pmatrix} B \\ C^T D \end{pmatrix}$$

$$C' = (D^T C \quad e^{-j\nu\pi} B^T)$$

$$D' = D^T D$$

- 1- H_∞ -Norm: Definition
- 2- Extension of Bounded Real Lemma
- 3- **Computation based on a derived fractional system stability analysis**

Derived fractional system computation

$\|G(s)\|_\infty < \gamma$ iff $(\gamma^2 I - G(-s)^T G(s))^{-1}$ is asymptotically stable

$$(\gamma^2 I - G(-s)^T G(s))^{-1} : \begin{cases} D^\nu x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \quad D_\gamma = (\gamma^2 I - D^T D)^{-1} \quad B_\gamma = \begin{pmatrix} B(D^T D - \gamma^2 I)^{-1} \\ C^T D(D^T D - \gamma^2 I)^{-1} \end{pmatrix}$$

$$A_\gamma = \begin{pmatrix} A + B(\gamma^2 I - D^T D)^{-1} D^T C & e^{-\nu j\pi} B(\gamma^2 I - D^T D)^{-1} B^T \\ C^T (I + D(\gamma^2 I - D^T D)^{-1} D^T) C & e^{-\nu j\pi} (A^T + C^T D(\gamma^2 I - D^T D)^{-1} B^T) \end{pmatrix} \quad C_\gamma^T = \begin{pmatrix} (\gamma^2 I - D^T D)^{-1} D^T C \\ e^{-\nu j\pi} (\gamma^2 I - D^T D)^{-1} B^T \end{pmatrix}$$

H_∞ -norm of **Fractional order system** given by

$$(S): \begin{cases} D^\nu x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is bounded by γ if and only if all the eigenvalues of A_γ lie in the stable domain defined by

$$\left\{ s \in \mathbb{C} : |\arg(s)| > \nu \frac{\pi}{2} \right\} \longrightarrow \boxed{\text{LMI}}$$

1- H_2 -Norm: Definition

2- Extension of Bounded Real Lemma

3- **Computation based on a derived fractional system stability analysis**

Summary

- Computation is performed through stability analysis of a derived system
- The method does not need any a priori knowledge of the bound
- More steps are necessary to obtain LMI formulation

4 – Conclusion and prospect

- Two methods:
 - Method 1: based an extension of the Bounded Real Lemma

Bound on H_∞ norm of fractional subsystem a priori known

- Method 2: based on stability analysis of a derived system

LMI Condition

- CRONE Control of LPV systems

Outline

Part 1: LMI tools for Stability Analysis of Fractional Systems

- 1 – LMI and integer order systems stability
- 2 – Stability of fractional order systems
- 3 – First method
- 4 – Second method
- 5 – Third method

Part 2: H_∞ -norm computation of fractional systems

- 1 – H_∞ -Norm: definition
- 2 – Extension of Bounded Real Lemma
- 3 – Computation based on a derived fractional system stability analysis

System definition

DC motor:
$$G(s) = \frac{K}{s(1 + \tau_e s)(f_m + J_m s)}$$

$$K = 2.34 \text{ N.m.V}^{-1}$$

$$\tau_e = 4.7 \times 10^{-3} \text{ s}$$

$$f_m = 2 \times 10^{-3} \text{ N.m.s.rad}^{-1}$$

$$J_m = J_0 + \Delta_J$$

$$J_0 = 0.066 \text{ kg.m}^2 \quad |\Delta_J| < 0.042 \text{ kg.m}^2$$

CRONE controller:
$$C(s) = K_c \left(\frac{s^\nu}{\omega_b} + 1 \right) / \left(\frac{s^\nu}{\omega_h} + 1 \right)$$

$$K_c = 2.82 \times 10^{-3}$$

$$\omega_b = 0.1 \text{ rad.s}^{-1}$$

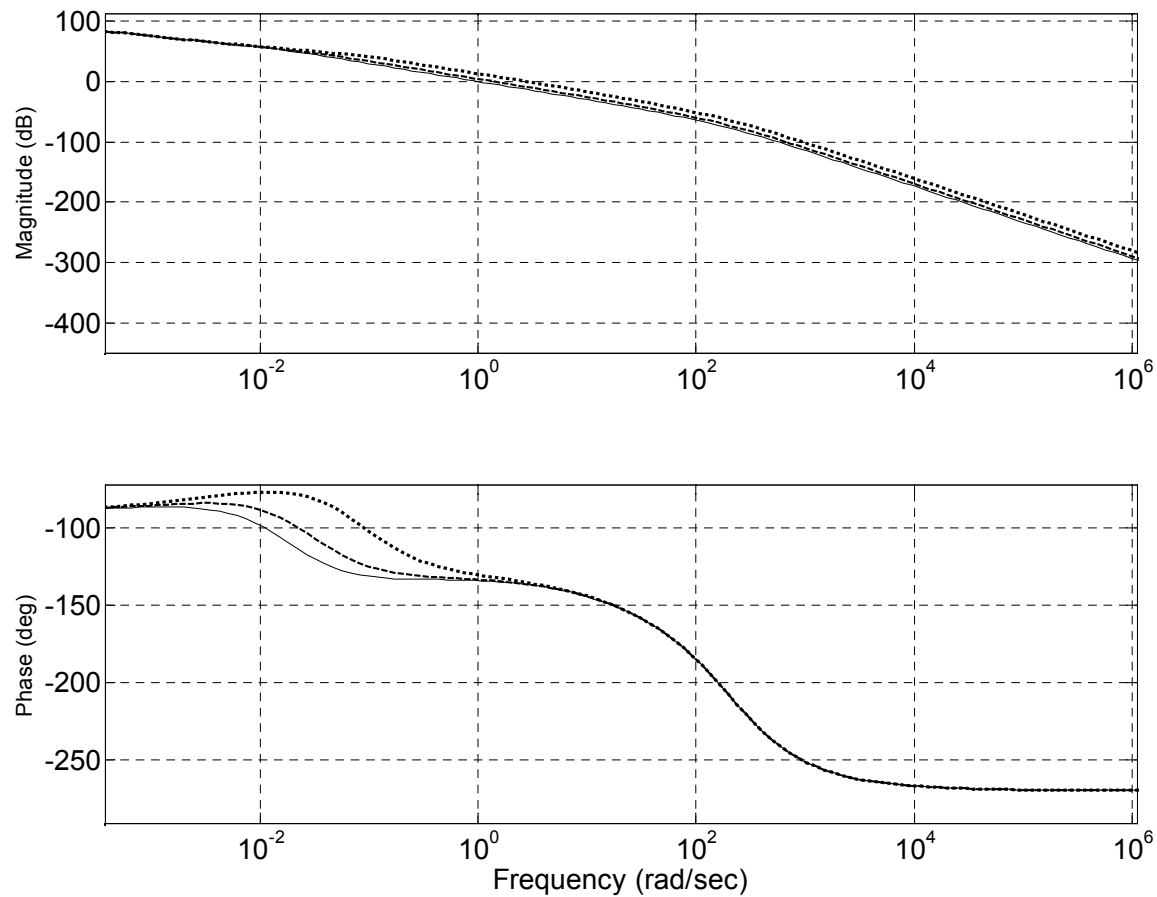
$$\omega_h = 10 \text{ rad.s}^{-1}$$

$$\nu = 0.6$$

To ensure: $\omega_{cg} = 1 \text{ rad.s}^{-1}$

System definition

Open loop:



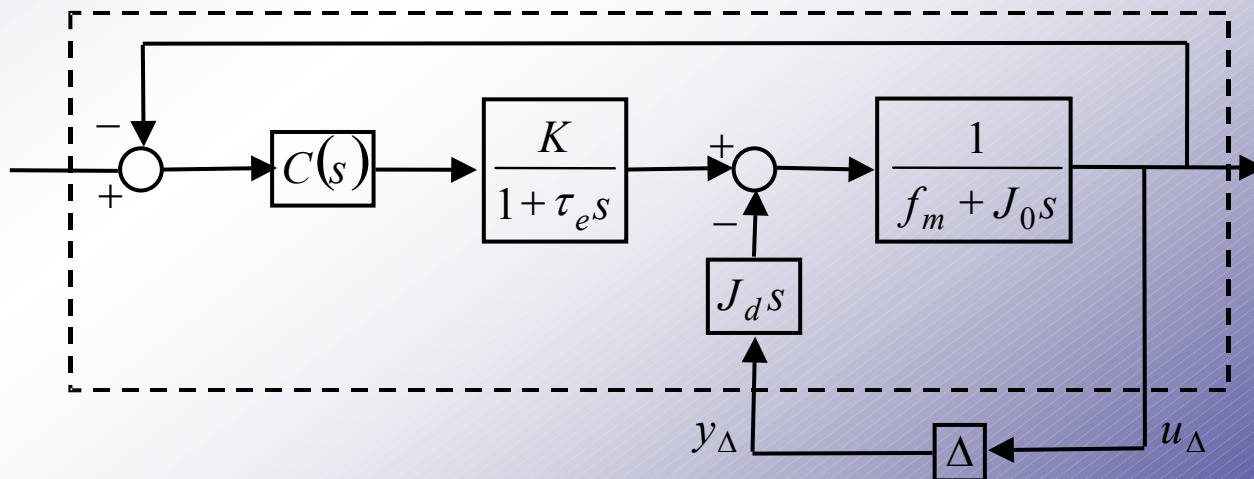
Stability analysis

$$G(s) = \frac{K}{s(1 + \tau_e s)(f_m + J_m s)}$$

Note that: $\frac{1}{f_m + J_m s} = \frac{1}{f_m + J_0 s} \frac{1}{1 + \frac{J_d s}{f_m + J_0 s}}$ with $J_d = 0.042$

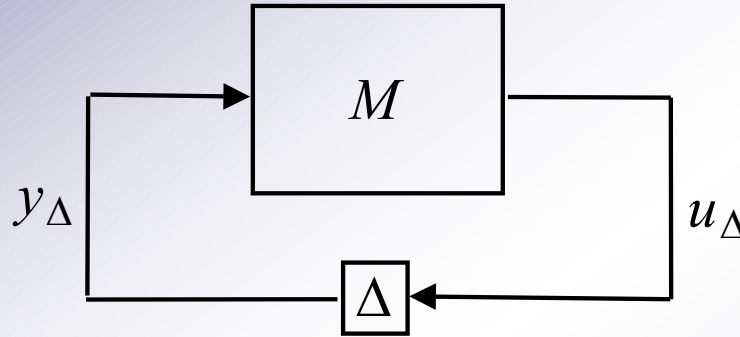
Thus $G(s) = \frac{K}{s(1 + \tau_e s)} \frac{1}{1 + \frac{\Delta_J s}{f_m + J_0 s}}$

Closed loop:



Stability analysis

Small gain theorem:



$$M(s) = \begin{pmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{pmatrix}$$

$$M_{22}(s) = \frac{-J_d s}{f_m + J_0 s} \left(\frac{1}{1 + CG_0} \right)$$

Stability condition: $\|M_{22}(s)\|_\infty \leq 1$

H_∞ norm computation : $\|M_{22}(s)\|_\infty < 0.882$

Thus closed loop is stable for $0.024 < J_m < 0.108$