UMR 7563

Autosimilarité de Processus Irréversibles

et Approche Non Entière en Viscoélasticité

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Les Systèmes à Dérivées Non Entière : Théorie et Applications

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eeigm

OS1 : Dynamique des fluides

- Structure des écoulements, stabilité, turbulence
- Ecoulements diphasiques
- Modélisation des systèmes complexes

OS2 : Mécanique et Energétique des fluides en situations réelles

- Thermorhéologie et mécanique des fluides complexes
- Mécanique et ingénierie cellulaire et tissulaire
- Turbulence et mécanismes de transferts dans les écoulements
- Echangeurs et systèmes thermiques et procédés

OS3 : Thermique et Mécanique des milieux hétérogènes

- Transferts thermiques
- Milieux poreux
- Mécanique du solide

OBJET D 'ETUDE

- RHEOLOGIE DES SOLIDES : CARACTERISATION MECANIQUE
  - VISCOELASTICITE LINEAIRE
    - Loi constitutive

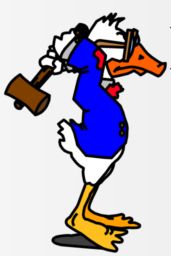
$$U(\sigma(t)) = \sum_{0}^{\infty} u_n \frac{d^n}{dt^n}[\sigma] = Q(\varepsilon(t)) = \sum_{0}^{\infty} q_m \frac{d^m}{dt^m}[\varepsilon]$$

Transformée de Laplace :

$$\bar{f} = L[f(t)] = \int_{0}^{\infty} f(t)\exp(-pt)\,dt \longrightarrow L\left[\frac{d^q f(t)}{dt^q}\right] = p^q \bar{f}(p)$$

condition initiale :
matériau au « repos »

$F(p)$ = relaxance



$\underline{E(p)}$
$\varepsilon(p)$ → **Système (matériau)** → $\underline{S(p)}$
$\sigma(p)$ $= F(p)\,E(p)$

Irreversibilité → Chaleur

ASPECTS SCIENTIFIQUES des PHENOMENES de RELAXATION

Fractal Geometry

MANDELBROT (1975)

Empirical scaling laws

decay $\phi(t) \propto \exp(-(t/\tau)^{\beta})$

$\phi(t) \propto (t/\tau)^{-\beta}$

KOHLRAUSCH (1863)
NUTTING (1921)
WILLIAMS & WATTS (1970)

DYNAMICS of RELAXATION

Le MEHAUTE, NIVANEN, NIGMATULLIN
SHLESINGER &KLAFTER
DISSADO & HILL
(1990)

Fractional Calculus
Non integer diff-integration operators

OLDHAM & SPANIER(1974)
OUSTALOUP (1990)

T.I.P

PRIGOGINE, COLEMAN, BIOT, GURTIN, NOWICK (1960-1970)

CUNAT (1990)

1. Modèles Non Entiers en Viscoélasticité

2. Modèle Basé sur la Thermodynamique des Processus Irréversibles (T.P.I.) - Approche DNLR (Dynamics of Non Linear Relaxations)

3. Un cadre théorique commun autour du concept de récursivité:

« HIERARCHICALLY CONSTRAINED DYNAMICS »

1. Modèles Non Entier en Viscoélasticité :

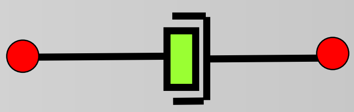
✦ MODELISATION ANALOGIQUE : le « SPRING-POT »

Le « SPRING »

Elément de Hooke

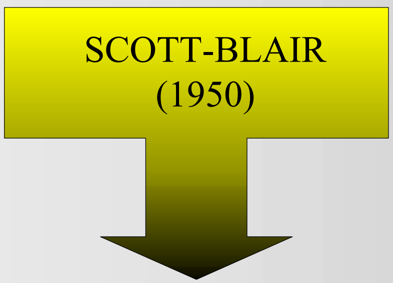
$$\sigma = E\,\varepsilon = E\,\frac{d^{(0)}\varepsilon}{dt^{0}}$$

Le « DASHPOT »

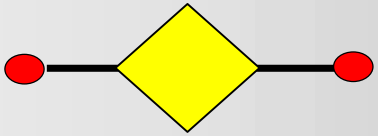


Elément de Newton

$$\sigma = \eta\,\dot{\varepsilon} = \eta\,\frac{d^{(1)}\varepsilon}{dt^{1}}$$



Le « SPRING POT »



$$\sigma = E\tau^{\alpha}\,\frac{d^{(\alpha)}\varepsilon}{dt^{\alpha}}$$

$$0 \leq \alpha \leq 1$$

solide — liquide

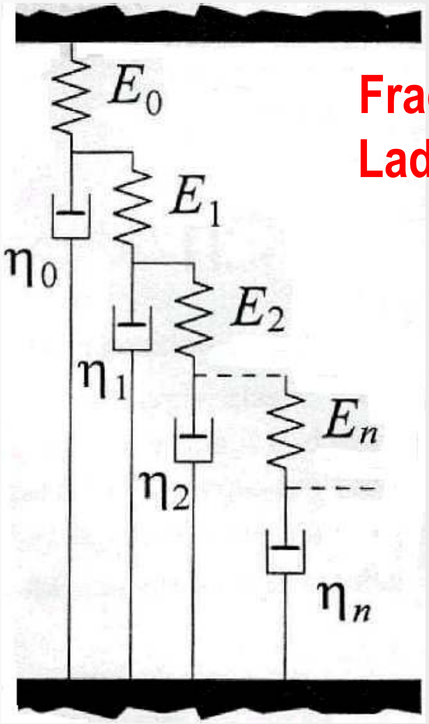
1. COMPLEXIFICATION par ASSEMBLAGES ANALOGIQUES

Elément fractionnaire réalisé par réseau 'itéré'

$$\sigma(t) = E\tau^{\alpha} \frac{D^{\alpha}\varepsilon}{Dt^{\alpha}}(t)$$

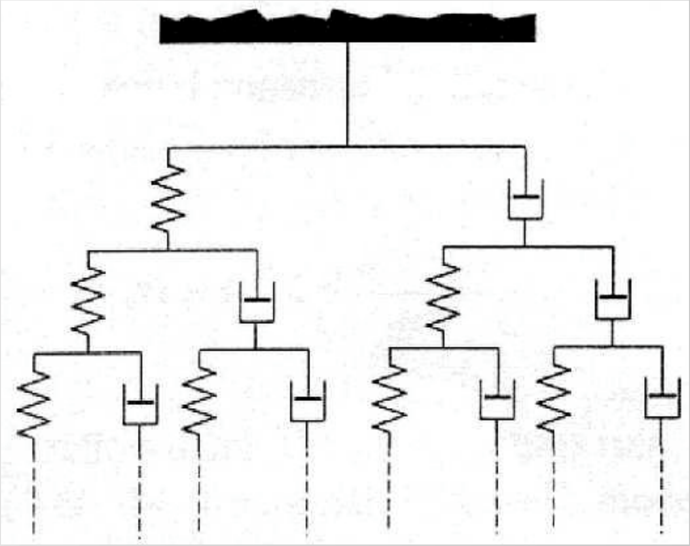
HEYMANS & BAWENS, 1994

α déterminé par des valeurs appropriées de E_i et η_i



Fractal Ladder

Fractal Tree



1. LOIS CONSTITUTIVES ' EMPIRIQUES '

- $$\sigma + \tau \frac{d^{\alpha}\sigma}{dt^{\alpha}} = E^{u}\tau \frac{d^{\beta}\varepsilon}{dt^{\beta}} + E^{r}\varepsilon \xrightarrow[\text{Laplace}]{\bar{f}(p) = L(f(t))} \bar{E} = \frac{E^{r} + E^{u}\tau p^{\beta}}{1 + \tau p^{\alpha}}$$

BAGLEY & TORVIK, 1985-1990

- **Modèles de relaxation empiriques (diélectriques)**

COLE & COLE, 1941 $$\bar{E} = \frac{1}{1 + \tau p^{\alpha}}$$

DAVIDSON & COLE, 1950 $$\bar{E} = \frac{1}{(1 + \tau p)^{\beta}}$$

$$\bar{E} = \frac{1}{(1 + \tau p^{\alpha})^{\beta}}$$

HAVRILIAK & NEGAMI, 1966

Nombre de paramètres restreint pour une reproduction excellente du comportement sur plusieurs décades

QUESTION OUVERTE: CONSISTANCE THERMODYNAMIQUE ?

1. MECANIQUE HEREDITAIRE

PRINCIPE de SUPERPOSITION de BOLTZMANN

Volterra's theory of integral equation : convolution integral form for past effects

$$\sigma(t) = E\left(\varepsilon(t) + \int_0^t f(t-\tau)\,\varepsilon(\tau)\,d\tau \right)$$

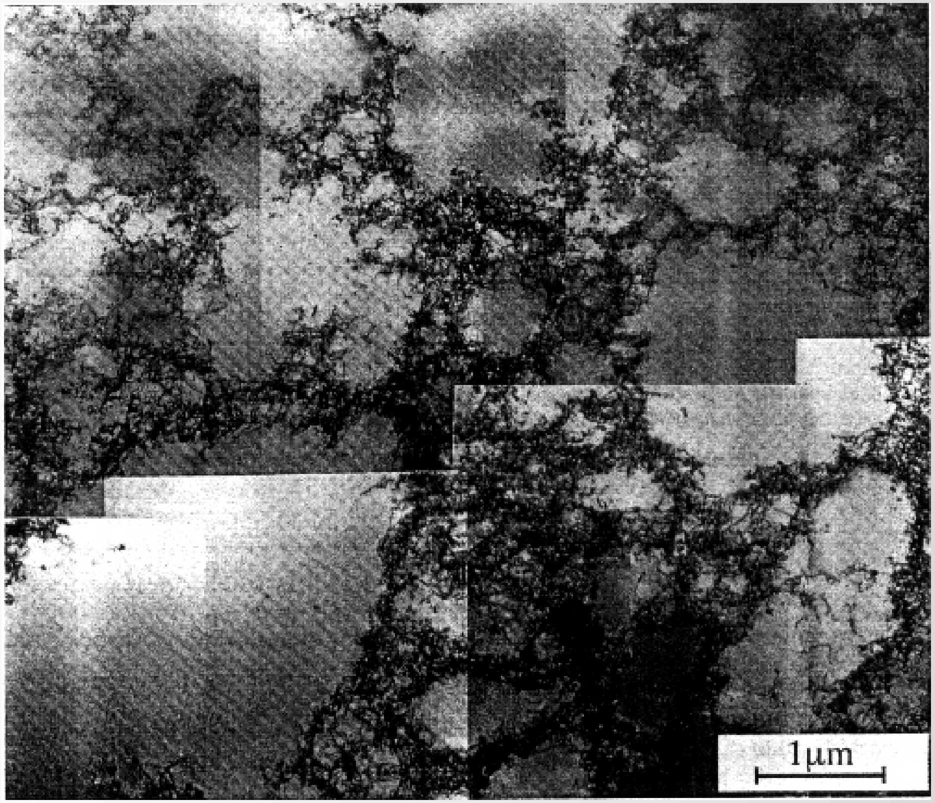
f : kernel or memory function

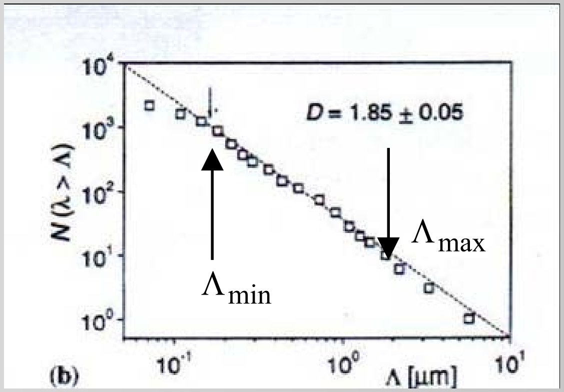
Singular kernel of type : $\dfrac{1}{\Gamma(1+\alpha)}(t-\tau)^{\alpha} \qquad -1 < \alpha < 0$

→ **Power law spectrum**

1. EVIDENCE of FRACTAL PATTERN in NATURE

FRACTAL DISLOCATION ARRANGEMENTS IN HEAVILY DEFORMED METALS





ZAISER & HÄHNER, 1999

Statistically self-similar spectrum

D : fractal dimension

1. ✦ L 'OPERATEUR de DIFF-INTEGRATION NON ENTIERE comme LIEN entre les DIFFERENTES APPROCHES

Le MEHAUTE, NIVANEN, NIGMATULLIN, 1990

Averaging procedure on a fractal (Cantor) set :

Empirical scale law kernel for memory effect representation

$$J(t) = K_{T,\nu}^{N^{\infty}} * f(t) = \int_0^t K_{T,\nu}^{N^{\infty}}(t-u) f(u) du = B(\nu)\frac{T^{-\nu}}{\Gamma(\nu)}\left(t^{-\nu} * f(t)\right) = B(\nu) T^{-\nu} D^{-\nu}\left[f(t)\right]$$

Cantor self-similar singularity distribution

C1 function

Theorem :

f : regularization kernel for the fractal set → **Riemann-Liouville integral operator (Non integer ν diff-integration)**

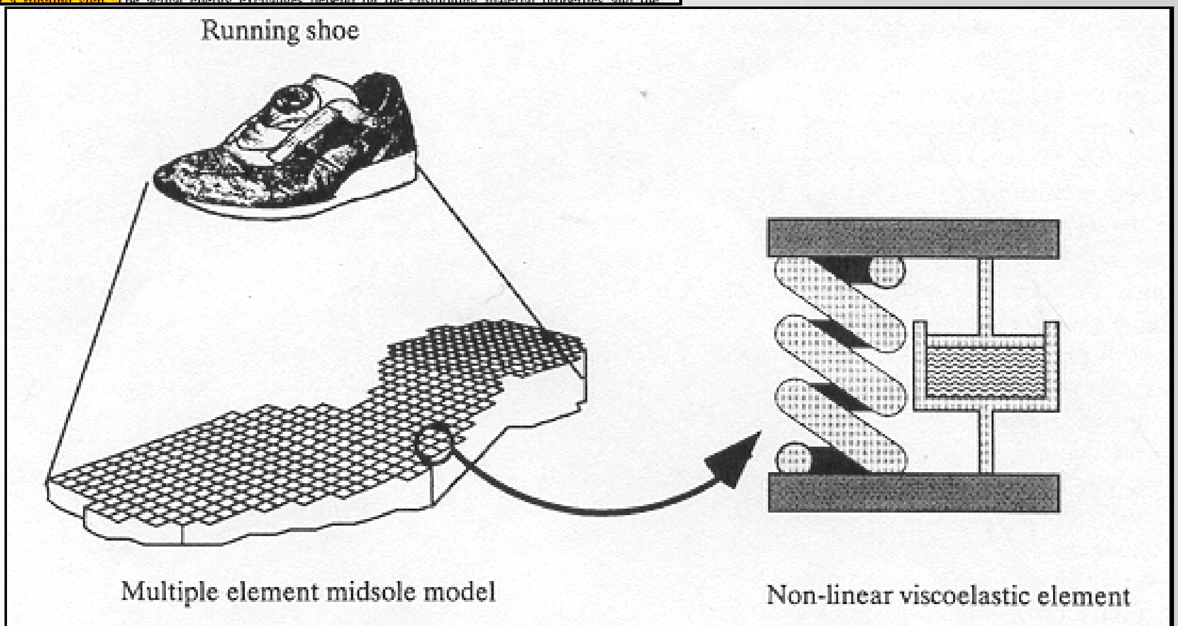
EXAMPLE IN ENGINEERING SCIENCE

THE ENERGETICS OF RUNNING AND RUNNING SHOES

MARTYN R. SHORTEN

2835 SE Tolman St., Portland, OR 97202-8752, U.S.A.

Abstract—It has been suggested that elastic energy storage and recovery in the cushioning system of an athletic shoe ('energy return') is a desirable quality that can enhance performance. However, comparing the energetics of a running shoe cushioning system with other passive energy exchange mechanisms in the running athlete suggests that the potential benefits of energy return are limited. The energetics of running shoe cushioning systems have been studied using a multiple-element, non-linear viscoelastic model to analyse the effect on the shoe of plantar pressure distributions recorded in vivo. The running shoe is a net dissipator of energy but small quantities of strain energy, of the order of 10J, are stored and recovered during a running step. The actual energy exchanges depend on the cushioning material properties and the



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2. MODELE D.N.L.R. basé sur la TPI

✦ DESCRIPTION du MODELE

$$\psi_k = \psi_k(\underline{\gamma}\,;\overline{z})$$

$$\begin{pmatrix} \dot{\beta} \\ -\dot{A} \end{pmatrix} = \begin{pmatrix} a & b \\ {}^t b & g \end{pmatrix} \begin{pmatrix} \dot{\gamma} \\ \dot{z} \end{pmatrix}$$

'Internal hidden reactions'

DE DONDER, 1920

$$\dot{z} = L\,A$$

ONSAGER, 1931

$$A = -\,g\,(z - z^r)$$

$$\dot{z} = -\frac{1}{\tau}(z - z^r)$$

MEIXNER, 1949

$$\tau = L\,g^{-1} = \begin{pmatrix} \diagdown & 0 \\ 0 & \diagdown \end{pmatrix}$$

MODAL ANALYSIS of DISSIPATIVE COMPONENT

D.N.L.R. spectrum

$$T\Delta S_i = \frac{1}{2}(z - z^r)g(z - z^r)$$

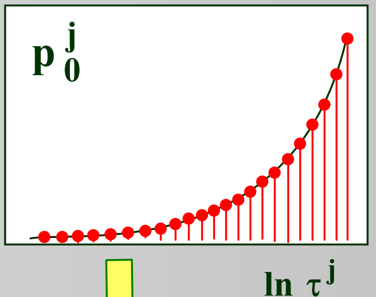
Fluctuation-dissipation theorem :

PRIGOGINE, 1968

Gaussian type relaxe state recovery

$$\langle \Delta S_i \rangle = -\frac{1}{2} r k_B$$

$$g^j \left\langle \left(z^j - z^{j,r}\right)^2 \right\rangle = k_B$$



$$p_0^j = B\sqrt{\tau^{j,r}}$$

2. FORMULATION PRATIQUE

$$\begin{pmatrix}\dot{\beta}\\ \dot{A}\end{pmatrix} = \begin{pmatrix} a & b\\ {}^{t}b & g\end{pmatrix}\begin{pmatrix}\dot{\gamma}\\ \dot{z}\end{pmatrix}$$

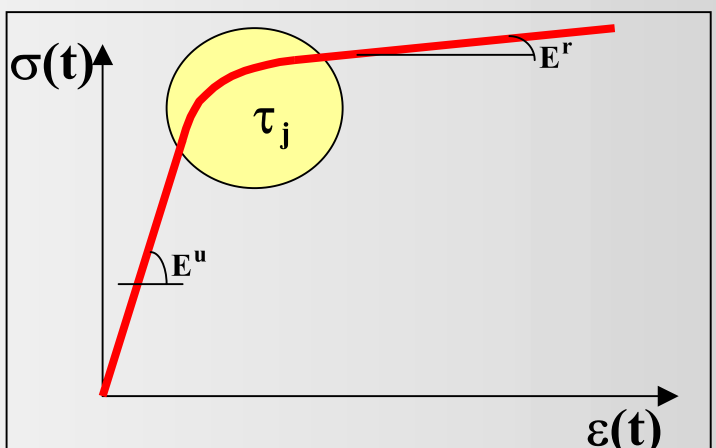
$$\begin{bmatrix}-\dot{S}\\ \dot{\sigma}\end{bmatrix} = \begin{bmatrix} -\dfrac{C^u_\varepsilon}{T} & -\alpha.E^u\\ -\alpha.E^u & E^u\end{bmatrix}\begin{bmatrix}\dot{T}\\ \dot{\varepsilon}\end{bmatrix} - \sum_j \begin{bmatrix} -\dfrac{S^j - S^{r,j}}{\tau^S_j}\\ \dfrac{\sigma^j - \sigma^{r,j}}{\tau^\sigma_j}\end{bmatrix}$$

OBSERVATION EQUATION

Test de traction isotherme 1D

$$\dot{\sigma} = \sum_{j=1}^{p}\dot{\sigma}^j = E^u\dot{\varepsilon} - \sum_{j=1}^{p}\frac{\sigma^j(t) - \sigma^{j,r}(t)}{\tau^j_\varepsilon} = \sum_{j=1}^{p} p^j_0 E^u\dot{\varepsilon} - \frac{\sigma^j(t) - \sigma^{j,r}(t)}{\tau^j_\varepsilon}$$

Tensile test, imposed $\dot{\varepsilon}$



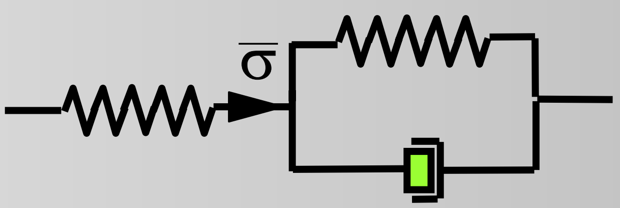
$$\sigma^{j,r} = E^r\,\varepsilon(t)$$

$$\sigma^{j,\bullet} = p^j_0\,\sigma^\bullet$$

2. REPRESENTATION ANALOGIQUE

1 seul processus = Zener Model

$$\sum \quad \dot{\sigma}^j(t) + \frac{\sigma^j(t)}{\tau_\varepsilon^j} = p_0^j\, E^u\, \dot{\varepsilon}(t) + \frac{p_0^j E^r \varepsilon(t)}{\tau_\varepsilon^j}$$

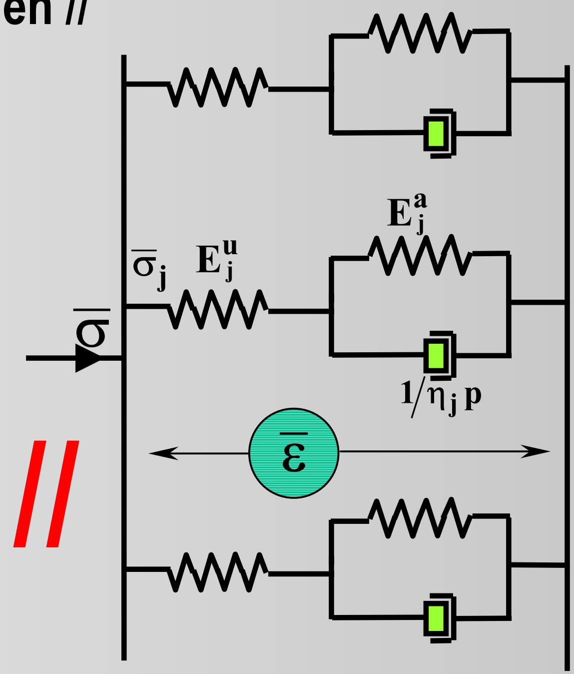


Plusieurs processus = Modèles de Zener en //

Transmittance :

$$\frac{\bar{\sigma}(p)}{\bar{\varepsilon}(p)} = E^u + \left(E^r - E^u\right)\sum_j \frac{p_0^j}{1+\tau^j p}$$

//



3. HIERARCHICALLY CONSTRAINED DYNAMICS

RECURSIVITE du MODELE DNLR

$$L\left\{ s_{imp}(t) = s(t) + \theta\, s(\phi\, t) + \theta^2\, s(\phi^2 t) + \ldots = \sum_j \theta^j\, s(\phi^j t) \right\}$$

$$\overline{S}(p) = L\left(s_{imp}(t)\right) = \sum_i \theta^i L\left(s(\phi^i t)\right) = \sum_i \left(\frac{\theta}{\phi}\right)^i \frac{\tau_i}{1+\tau_i p}$$

DNLR Admittance (anélastique)

$$Y^*(p) = \sum_j \frac{p_0^j\, p}{1+\tau^j p}$$

- **Temporal recursivity : logarithmically equally spaced times**

$$\tau_0 = \phi^j \tau_j \qquad \tau_j = \phi\ \tau_{j+1}$$

$$\phi = 10^{\frac{\delta}{N-1}}$$

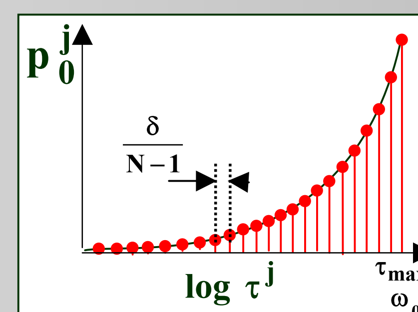
- **Equipartition of entropy rate ⇒ Magnitude recursivity**

$$p_0^j = B\sqrt{\tau^{j,r}} \Longrightarrow p_0^j = (\phi)^{1/2}\ p_0^{j+1}$$

- **Non-integer characteristic number :**

$$\theta = \phi^{3/2}$$

$$n = 1 - \frac{\log\theta}{\log\phi} = 0{,}5$$



Number of decades δ

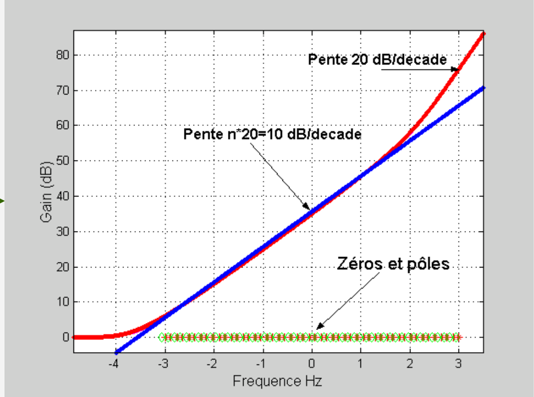
3. DNLR MODEL and FRACTIONAL CALCULUS

DNLR Admittance

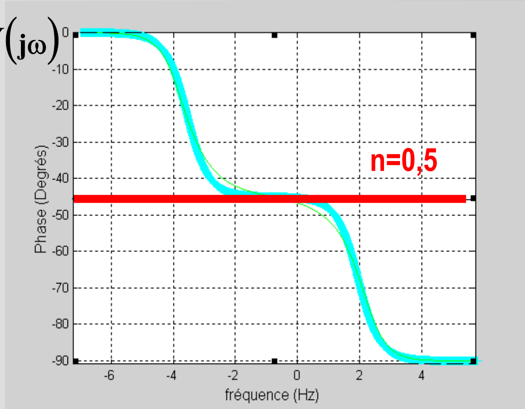
$$Y^*(j\omega) = \sum_j \frac{p_0^j \, j\omega}{1 + j\omega\tau^j}$$

BODE DIAGRAMS

$|Y(j\omega)|$



$\mathrm{Arg}\,Y(j\omega)$



SIGNATURE d 'un OPERATEUR DIFFERENTIEL NON - ENTIER

$$\beta(t) = \tau^n \frac{d^n}{dt^n} \gamma(t)$$ → **Fractional « spring-pot »**

$$\bar{\beta}(p) = (\tau p)^n \, \bar{\gamma}(p) \quad \Rightarrow \quad Y(p) = (\tau p)^n$$

$$p = j\omega$$

$$|Y(j\omega)| = (\tau\omega)^n$$

$$\mathrm{Arg}(Y(j\omega)) = n\frac{\pi}{2}$$

OLDHAM, ZOSKI, 1983

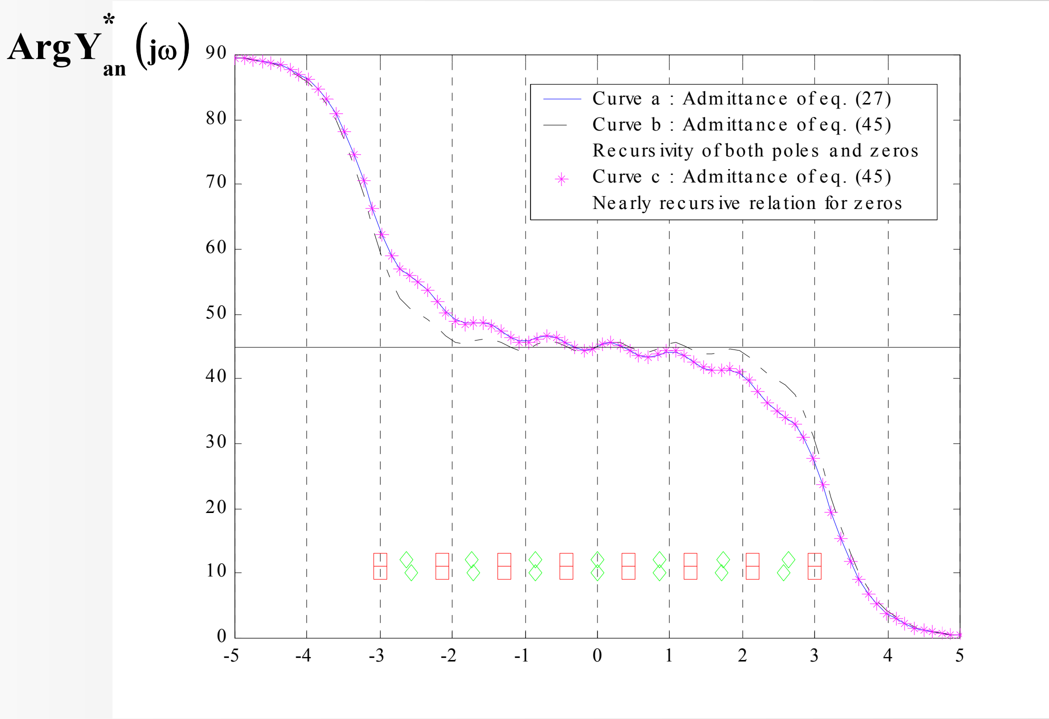
Non borné en fréquence

3. DISTRIBUTION de 0 et de POLES : APPROCHE de BIOT (1950)

$$Y^*(j\omega) = \sum_j \frac{p_0^j\, j\omega}{1 + j\omega\tau^j}$$

$$Y^*_{an}(s) \cong Y_N(s) = \frac{s}{P^0} \, \frac{\prod_{j=1}^{N-1}\left(1+\frac{s}{\omega'_j}\right)}{\prod_{j=1}^{N}\left(1+\frac{s}{\omega_j}\right)}$$

$$1/P^0 = \sum_j p_j^0 = 1$$



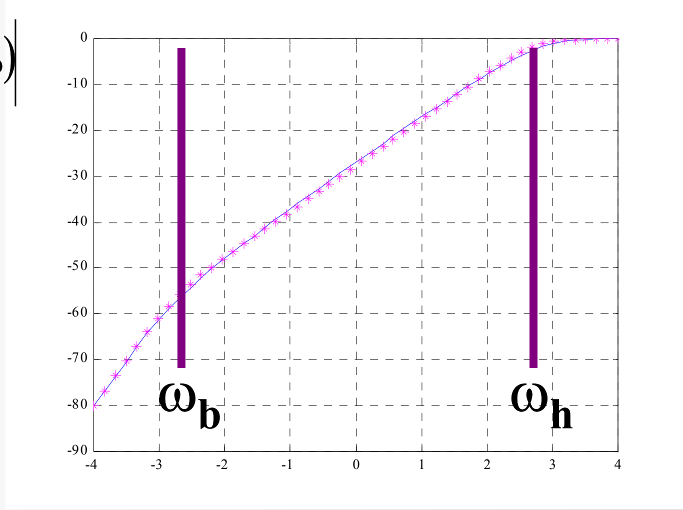
PERTE de RECURSIVITE sur les ZEROS !!!

3. MODELE de OUSTALOUP (1995)

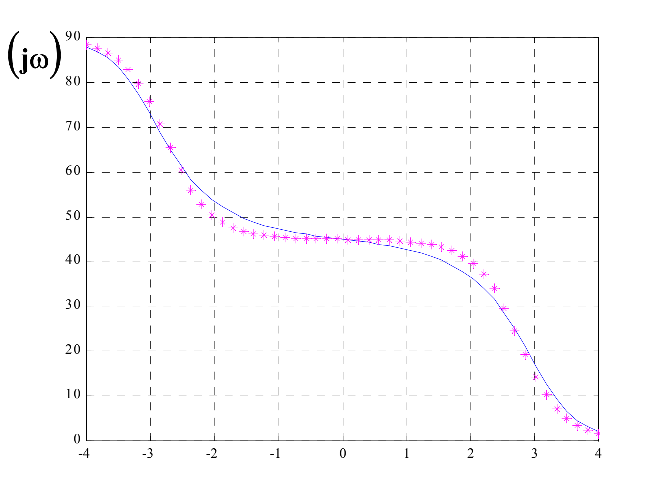
$$Y^*_{an}(s) = Y_{n'}(s) = s\,\frac{\left(1+\dfrac{s}{\omega_h}\right)^{n'-1}}{\left(1+\dfrac{s}{\omega_b}\right)^{n'}}$$

MODELE BORNE en FREQUENCES

$\left|Y^*_{an}(j\omega)\right|$



$\mathrm{Arg}\,Y^*_{an}(j\omega)$



—— **Modèle DNLR**

* * * * * **Modèle ‘ fractionnaire ’ (Oustaloup)**

3. MODELE de OUSTALOUP (1995) - CARACTERISATION TEMPORELL

$$\left(\frac{d}{dt}\right)^{n}_{impl,\tau} f(t) = \left(\frac{d}{dt}\right)^{n} \left[f(t)\exp(t/\tau)\right]$$

Dérivation non entière implicite

Loi constitutive

$$\varepsilon(t) = s^{u}\,\sigma(t) + s^{a}\,\omega_h^{1-n}\,e^{-\omega_h t}\left(\frac{d}{dt}\right)^{n-1}_{impl,\omega_h}\left[\omega_b^{n}\,e^{-\omega_b t}\left(\frac{d}{dt}\right)^{-n}_{impl,\omega_b}\sigma(t)\right]$$

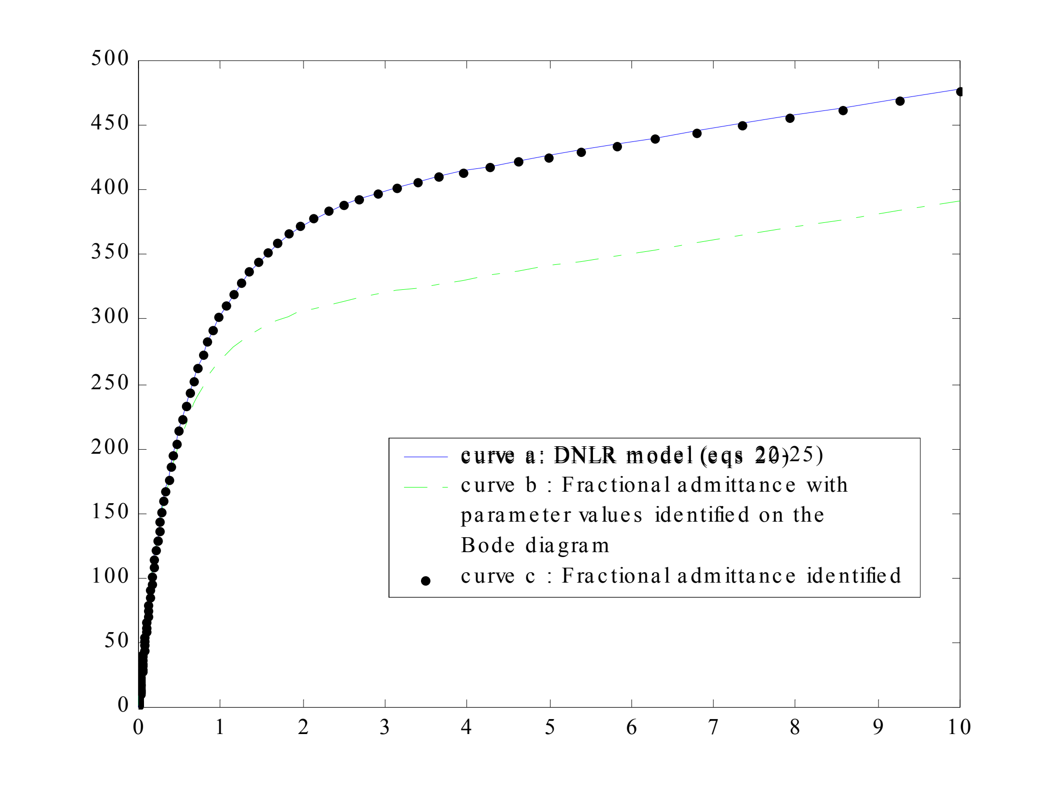
Forme complexe !

MAIS :

- **Introduction empirique par Friedrich et al. (1999)**

- **Offset fractional diffintegration operator introduced by LeMahauté et al. (Exponentially damping of a fractal Cantor set distribution)**

3. MODELE de OUSTALOUP (1995) - SIMULATION ESSAI TRACTION



CALCUL : **Retour numérique Laplace**

(algorithmes de Stehfest, Hoog, Tfourier)

PRECIS et RAPIDE

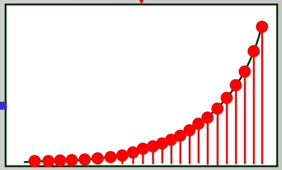
CONCLUSION

TPI DNLR Model = Modal Analysis of dissipation

- **DISTRIBUTION of NON LINEAR RELAXATIONS**

  POSTULAT 1 : Temporal recursivity

- **THEORY OF FLUCTUATIONS**



CONSEQUENCE 1 : Recursivity in magnitude

CONSEQUENCE 2 : Existence of a non-integer order of differentiation

Memory functions on continuous variable ⟷ Discrete hierarchical dynamical systems

CONSEQUENCE 3 ??: LINK with UNDERLYING RECURSIVE GEOMETRY (self-similar patterns) of the MICROSCOPIC STRUCTURE

↓

Cluster Model of Dissado & Hill (1985)