MODELLING AND SIMULATION OF FRACTIONAL SYSTEMS USING A NON INTEGER INTEGRATOR

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ABSTRACT

An original method for modelling and simulation of fractional systems is presented in this paper. The basic idea is to model the fractional system by a state-space representation, where conventional integration is replaced by a fractional one with the help of a non integer integrator. This operator is itself approximated by a $N+1$ dimensional system composed of an integrator and of a phase-lead filter. This method is compared to other techniques like direct discretization of the fractional derivator and diffusive representation. Numerical simulations exhibit the general applicability and flexibility of this new approach to different types of fractional models and to non conventional non integer derivation with limited spectral range.

INTRODUCTION

Non integer order systems, also known as fractional filters, have been introduced long ago in various fields of science such as electrochemistry [1], acoustics [2], electromagnetism [3] ... where they are fundamentally used for the modelling of diffusion processes. These systems are characterized by long memory transients and infinite dimensional structure. Their dynamics depend on the well-known Diffusion Equation and on the geometry of the considered problem. If the attention is focussed on the relation between variables at the boundary region, a theoretical modelling leads to an integrator with order equal to 0.5. Generalization of this modelling to more complex situations needs the help of a fractional model, characterized by its non integer order, whose value can vary from 0 and 1. Practically, the modelling of fractional systems turns out to be very usefull for simulation, identification and control [4–8].

This modelling is fundamentally based on non integer derivation. Then, the numerical simulation of these systems is highly linked to the modelling of the non integer derivator, or equivalently of the non integer integrator. Two main approaches are commonly used. A direct solution [6] is based on the discretization of the derivator: then, the fractional model is replaced by a difference equation model, with long memory behaviour. An indirect solution uses Diffusive Representation [9, 10], with finite discretization of the continuous fractional model into a N dimensional state-space representation with conventional integer derivation.

In this paper, we propose a third approach based on a fractional integration operator in order to simulate a fractional system with a conventional state-space representation approach. Fundamentally, simulation of a state-space model needs integration operators ; thus simulation of a fractional model needs an equivalent non integer integration operator. This new operator is defined in the frequency domain, with reference to the ideal integrator of order n ($0 < n < 1$). An approximation is necessary, because it is impossible to use an infinite spectral range. Thus the corresponding fractional operator acts with order n on a limited frequency range and with order one outside it. Moreover, this approximation is performed by a finite dimensional system or equivalently by its finite state-space representation. Then, this fractional integration operator is used to simulate the corresponding system with an appropriate macro state-space representation.

This paper is divided in four parts. The first one presents the main methods used in order to simulate fractional systems. The second part is devoted to the definition and the modelling of the fractional integrator. This new operator is used to define the state-space representation of a non integer system in the third part. Modelling of different types of fractional systems are illustrated by numerical simulations in the fourth part.

SIMULATION OF FRACTIONAL SYSTEMS

Many methods are used in order to simulate non integer or fractional systems. Two types of methods can be considered. The first ones, also called direct methods, are based on an numerical approximation of the non integer derivator operator. The second ones, called indirect methods, are based on the simulation of the continuous fractional model, with the help of a specific operator or representation.

Direct methods

In these methods the fractional derivator operator is replaced by a numerical approximation, in order to obtain a recurrent equation directly used for simulation. Different types of approximations can be used. The more usual approximation is the one directly related to the definition given by Grünwald [11]:

$$\frac{d^n}{dt^n} f(Kh) = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} f((K-k)h) \qquad (1)$$

where h is the sampling period and:

$$\binom{n}{k} = \frac{n\,(n-1)\,(n-2)\,\cdots\,(n-k+1)}{k!} \qquad (2)$$

In order to illustrate this approach, consider for instance the fractional system defined by:

$$\frac{d^n y(t)}{dt^n} + a_0 y(t) = b_0 u(t) \quad 0 < n < 1 \qquad (3)$$

which system will be used as a benchmark model for the different techniques of simulation.

Using previous numerical approximation, the following equation is obtained [6]:

$$\sum_{k=0}^{K} \frac{(-1)^k}{h^n} \binom{n}{k} y((K-k)h) + a_0 y(Kh) = b_0 u(Kh) \qquad (4)$$

where K is the number of data such as $t = Kh$.

The system output is given by:

$$y(Kh) = \frac{b_0 u(Kh) - \sum_{k=1}^{K} \frac{(-1)^k}{h^n} \binom{n}{k} y((K-k)h)}{a_0 + \frac{1}{h^n}} \qquad (5)$$

This method is very simple to use. On the other hand, the simulation requires, for each step, the computation of sums of increasing dimension with time. Other approximations can be used, refer for instance to [12].

Indirect method based on diffusive representation

Indirect methods refer to the simulation of the continuous fractional model with the help of a specific operator or representation. Diffusive Representation (DR) approach [9, 10] is typically one of these methods.

Consider a fractional system, whose impulse response is $h(t)$: it is composed of a weighted sum of an infinity of modes $e^{-\xi_k t}$, where ξ_k varies from very small to very large values.

Thus

$$h(t) = \sum_{k=1}^{K} \mu(\xi_k)\; e^{-\xi_k t}\; \Delta\xi_k \qquad (6)$$

Considering $\Delta\xi_k \to 0$ and $K \to \infty$, the continuous limit is:

$$h(t) = \int_0^{\infty} \mu(\xi)\; e^{-\xi t}\; d\xi \qquad (7)$$

The function $\mu(\xi)$ is the weight density of $e^{-\xi t}$ modes, it is called Diffusive Representation (DR).

Notice that in equation (7) $h(t)$ can be interpreted as the transform (Laplace transform L) of $\mu(\xi)$.

Thus, $\mu(\xi)$ can be obtained from $h(t)$ using inverse transform (L^{-1}):

$$\mu(\xi) = L^{-1}\{h(t)\} \qquad (8)$$

An infinite dimension state vector representation is attached to each fractional system.

Let $u(t)$ be the input and $y(t)$ the output ; then the input/output representation of the system is given by:

$$y(t) = h(t) * u(t) \qquad (9)$$

This input/output representation (9) is equivalent to the following state vector model:

$$\frac{\partial x(\xi,t)}{\partial t} = -\xi x(\xi,t) + u(t) \qquad (10)$$

$$y(t) = \int_0^{\infty} \mu(\xi)\; x(\xi,t)\; d\xi \qquad (11)$$

Practically, numerical simulation of a fractional system is based on discretization of the variable ξ in equations (10) and (11):

$$\frac{dx_k(t)}{dt} = -\xi_k x_k(t) + u(t) \quad (k = 1\, to\, K) \tag{12}$$

$$y(t) = \sum_{k=1}^{K} \mu(\xi_k)\; x_k(t)\; \Delta\xi_k \tag{13}$$

Consider previous benchmark model (3):

$$H(s) = \frac{b_0}{a_0 + s^n} \quad 0 < n < 1 \tag{14}$$

It is necessary to use the DR $\mu(\xi)$ corresponding to this model. G. Montseny has demonstrated [10] that:

$$\mu(\xi) = \frac{b_0 \frac{\sin(n\pi)}{\pi} \xi^n}{\xi^{2n} + 2a_0 \frac{\cos(n\pi)}{\pi} \xi^n + a_0^2} \tag{15}$$

The next step is the choice of the discretization for ξ_k. A geometric progression for ξ_k and $\Delta\xi_k$ is certainly the best choice for this approximation. Thus we obtain the system (12) and (13) approximating the fractional model.

A FRACTIONAL INTEGRATOR OPERATOR

Fractional derivator

There are many ways to define the ideal fractional derivator [6]. Using the transform technique, let $D(s) = s^n$ be the Laplace transform of $\left(\frac{d^n}{dt^n}\right)$, where $0 < n < 1$. Then $D(j\omega)$ is the Fourier transform defined by:

$$D(j\omega) = \rho e^{j\theta} \qquad \rho = \omega^n, \theta = n\frac{\pi}{2} \tag{16}$$

With this definition, the fractional domain where $D(j\omega)$ acts as a non integer derivator ranges from 0 to ∞. A. Oustaloup [6,7] has shown that the spectral range has to be necessarily limited to $[\omega_b, \omega_h]$.

Moreover, experiments show that this spectral range can be very limited, for instance over one decade when behaviour is caused by an artificial diffusion process [6]. On the contrary, when the domain $[\omega_b, \omega_h]$ is very large, there is not a great difference with an ideal derivator. Thus, for narrow band derivators, it is necessary to define in a realistic way the behaviour of the derivator outside the limited domain.

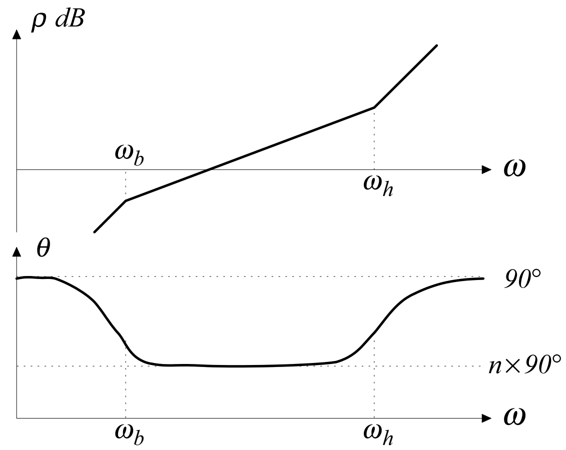

Figure 1. BODE DIAGRAM OF THE FRACTIONAL DERIVATOR

So, let us consider the Bode plot of the proposed practical derivator on figure 1.

It is composed of three parts:

The intermediate part corresponds to non integer action, characterized by the order n $(0 < n < 1)$.
In the two other parts, the derivator has conventional action, characterized by its order 1.

In this way, we define a new operator $D_n(j\omega)$ $(D_n(s))$ which is a conventional derivator, excepted on a limited band $[\omega_b, \omega_h]$ where it acts like a "n" non integer derivator. Thus, this operator $D_n(s)$ is characterized by three parameters : ω_b, ω_h and n.

Fractional integrator

Practically, simulation of systems is not performed with derivators but with integrators. It is easy to define the practical non integer integrator as the inverse of $D_n(s)$, thus

$$I_n(s) = \frac{1}{D_n(s)} \tag{17}$$

So, our interest will be focussed on the modelling and simulation of the operator $I_n(s)$, whose Bode plot is dual of that of $D_{j\omega}(s)$. This Bode diagram can be obtained using a transfer function like:

$$I_n(s) = \frac{G_n}{s}\left(\frac{1 + \frac{s}{\omega_b}}{1 + \frac{s}{\omega_h}}\right)^n \tag{18}$$

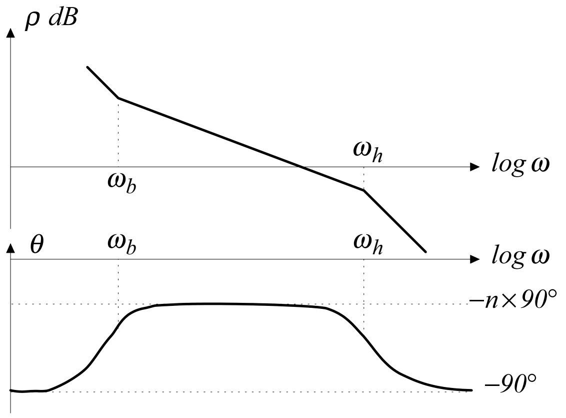

Figure 2. BODE DIAGRAM OF THE FRACTIONAL INTEGRATOR

The synthesis of this operator is performed by the association of an integrator $\frac{1}{s}$ and of the conventional phase lead filter used by A. Oustaloup [6]:

$$A_v(j\omega) = \prod_{i=1}^{N} \frac{1 + j\frac{\omega}{\omega'_i}}{1 + j\frac{\omega}{\omega_i}} \qquad (19)$$

composed of N cells and characterized by four design parameters $\omega_i^{'}$, ω_i, α, η and n'. Let us define:

$\omega_1^{'}$: the lower pulsation,
ω_N: the higher one.

with, for the i^{th} cell:

$$\begin{aligned} \omega_i &= \alpha\,\omega_i^{'} \quad with \quad \alpha > 1 \\ \omega_{i+1}^{'} &= \eta\,\omega_i \quad with \quad \eta > 1 \end{aligned}$$

If N is sufficiently high, the Bode plot of $A_v(j\omega)$, inside $\left[\omega_1^{'}, \omega_N\right]$, is characterized by a positive slope equal to $n' \times 20dB/dec$ and a constant positive phase equal to $n' \times 90^o$, where

$$n^{'} = \frac{\log\alpha}{\log\alpha\eta} \qquad (20)$$

Thus, combining $A_v(j\omega)$ with conventional integrator $\frac{1}{j\omega}$, we obtain the Bode plot of figure 2.

This means that the operator $I_n(s)$ can be approximated by:

$$I_n(s) = \frac{G_n}{s} \prod_{i=1}^{N} \frac{1 + \frac{s}{\omega'_i}}{1 + \frac{s}{\omega_i}} \qquad (21)$$

This operator is characterized by six parameters:

$\omega_1^{'}$ and ω_N define the frequency range (equivalently to ω_b and ω_h),
N is the number of cells (it is directly related to the quality of the desired approximation),
α and η are recursive parameters related to non integer order n.
G_n is defined in order to have the same gain for $\frac{1}{s^n}$ and $I_n(s)$ at the pulsation $\omega_u = 1\,rd/s$

This operator is completely defined by the following relations:

$$\omega_i = \alpha\,\omega_i^{'}, \quad \omega_{i+1}^{'} = \eta\,\omega_i, \quad n = 1 - \frac{\log\alpha}{\log\alpha\eta} \qquad (22)$$

State-space representation of the operator

It is convenient to associate a state-space representation to $I_n(s)$ in order to simulate complex systems. An infinity of these representations can be associated to $I_n(s)$. Because one of our objectives is to estimate the parameters $\{\omega_1^{'}, \omega_N, \alpha \text{ and } \eta\}$, we have privileged parsimonious models in order to facilitate the identification procedure [4, 5, 8].

Operator $I_n(s)$ can be expanded using ω_i pulsations:

$$I_n(s) = \frac{G_n}{s} \prod_{i=1}^{N} \frac{1 + \frac{s}{\omega'_i}}{1 + \frac{s}{\omega_i}} = \frac{G_n}{s} + \sum_{i=1}^{N} \frac{c_i}{s + \omega_i} \qquad (23)$$

where

$$c_i = G_n \frac{\omega_i - \omega'_i}{\omega'_i} \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1 - \frac{\omega_i}{\omega'_j}}{1 - \frac{\omega_i}{\omega_j}} \qquad (24)$$

Thus, it is straightforward to use the following state-space model to simulate the fractional integrator:

$$\frac{d}{dt}\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & -\omega_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\omega_N \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}}_{\underline{x_I}} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{\underline{B}} u(t) \qquad (25)$$

$$x_{\text{int}} = \underbrace{\begin{bmatrix} G_n & c_1 & \cdots & c_N \end{bmatrix}}_{\underline{C}^T} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} \qquad (26)$$

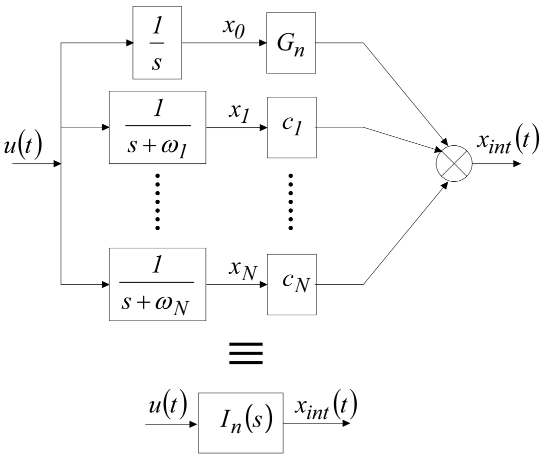

Figure 3. STATE-SPACE DECOMPOSITION OF THE FRACTIONAL INTEGRATOR

equivalent to

$$\begin{cases} \frac{d}{dt}\underline{x}_I = A\,\underline{x}_I + \underline{B}\,u \\ x_{\text{int}} = \underline{C}^T\,\underline{x}_I \end{cases} \tag{27}$$

where the state-variables x_i are the outputs of the first order model blocks $\frac{1}{s+\omega_i}$ (see figure 3). Notice that the first output is x_0, where $\omega_0 = 0$ and "weight" $c_0 = G_n$.

STATE-SPACE REPRESENTATION OF A NON-INTEGER SYSTEM

Principle

Using the fractional integrator, one can construct the state-space representation of general fractional systems. We will again consider the benchmark model (3):

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{a_0 + s^n} \tag{28}$$

This transmittance is equivalent to the differential equation form:

$$\frac{d^n y(t)}{dt^n} + a_0 y(t) = b_0 u(t) \tag{29}$$

Let us define $x(t)$ such as

$$X(s) = \frac{1}{s^n + a_0} U(s) \tag{30}$$

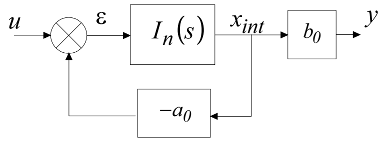

Figure 4. STATE-SPACE REPRESENTATION OF THE SYSTEM

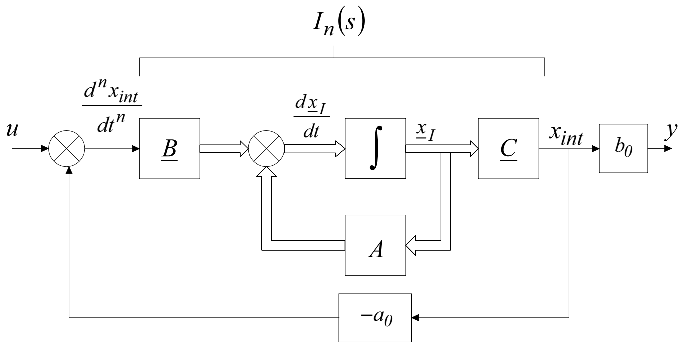

Figure 5. THE TWO IMBRICATED STATE-SPACE MODELS

Thus, we obtain a "macro" state-space representation of this system (with "macro" parameters a_0 and b_0).

$$\begin{cases} \frac{d^n x(t)}{dt^n} = -a_0 x(t) + u(t) \\ y(t) = b_0 x(t) \end{cases} \tag{31}$$

where $x(t) = x_{int}(t)$ and $\varepsilon(t) = u(t) - a_0 x_{int}(t)$ is the input of $I_n(s)$ (see figure 4).

Analysis of the state-space model

This "macro" model (31) is only convenient for compact writing. Practically, there are two imbricated state-space representations, one for the "macro" model, the other one for the fractional integrator, as represented on figure 5.

Remarks:

N has to be very large in order to perform an appropriate approximation of the fractional integrator. Then, $\underline{x}_I$ is a large dimension vector, able to produce long memory behaviour [13], the main feature of fractional systems.

The differential equation is characterized by fractional order n. This parameter is not explicitly used in the model, because it has been converted into four equivalent parameters ω_1', ω_N, α and η. This transformation enables us to simulate the fractional system with a conventional equivalent state-space model and so to estimate its parameters in

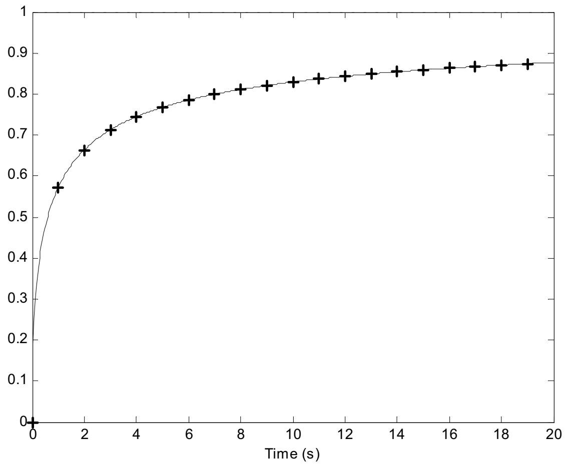

Figure 6. STEP RESPONSE FOR $n = 0.5$ - (-: SIMULATION WITH $I_n(s)$, +: SIMULATION WITH THE DIRECT METHOD)

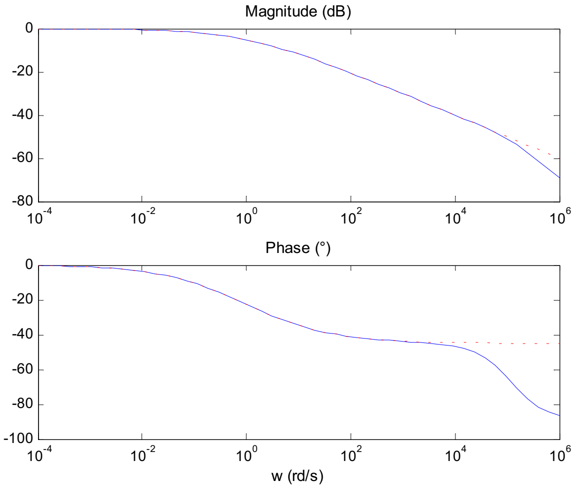

Figure 7. BODE PLOT FOR $n = 0.5$ AND $N = 30$ - (-: APPROXIMATED SYSTEM, .: THEORETICAL SYSTEM)

identification applications [4,5,8]. With α and η estimates it is possible to derive n, using $n = 1 - \frac{\log\alpha}{\log\alpha\eta}$
Different state-space representations have been derived to simulate fractional systems. A. Oustaloup [6] has proposed an infinite dimensional model based on the numerical approximation of the non integer derivator. Recently, new models have been introduced; refer for instance to [14, 15].

NUMERICAL SIMULATIONS

Benchmark fractional system

Simulations. The modelling of the operator $I_n(s)$ permits to approach the ideal fractional integrator when the frequency range is very large. For a fractional order n equal to 0.5, step responses of the benchmark fractional system (3) have been simulated using the operator $I_n(s)$ and the direct method described in the first section.

Figure 6 shows step responses obtained by these two methods with:

$a_0 = 1$,
$b_0 = 1$,
$\omega_1' = 10^{-5} rd/s$,
$\omega_N = 10^5 rd/s$
a number of cells $N = 30$.

One can notice that step responses are superposed.

The frequency responses of the theoretical fractional system and its approximation using $I_n(s)$ are plotted figure 7. One can notice that inside the frequency domain defined by ω_1' and ω_N, the two responses coincide. Beyond this domain, the system is equivalent to a first order system.

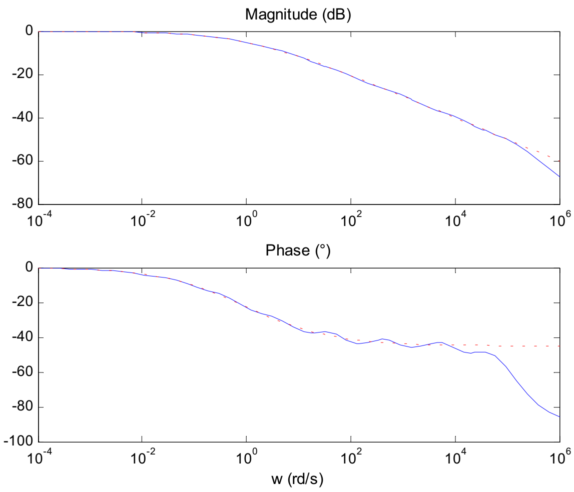

Figure 8. BODE PLOT FOR $n = 0.5$ AND $N = 10$ - (-: APPROXIMATED SYSTEM, .: THEORETICAL SYSTEM)

For a lower value of N ($N = 10$), the frequency response is plotted on figure 8. One can notice the phase oscillations directly linked to the insufficient number of cells.

Conclusion. The direct method gives satisfactory results for any value of n. We obtain the same result with $I_n(s)$ when the number of cells is sufficiently large. We have not presented simulation results for the Diffusive Representation approach but as expected, they are also satisfactory.

The direct approach applies to any kind of fractional sys-

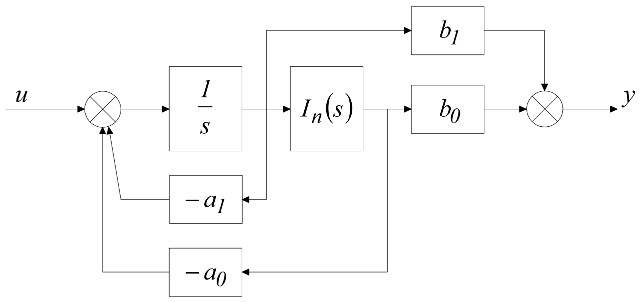

Figure 9. STATE-SPACE REPRESENTATION OF THE SYSTEM (32)

tems: its essential drawback is the increasing dimension of the sums directly related to the numerical approximation (1). This is a real constraint when this approximation is used to simulate the model in identification and parameter estimation.

The Diffusive Representation needs an analytic knowledge of $\mu(\xi)$, which is a drastic constraint in practice for complex systems.

Simulation of a fractional system with bounded spectral range

When the frequency domain where acts the non integer order is not infinite but limited to some decades, the operator defined in this paper permits nevertheless to give satisfactory time and frequency responses.

Consider the system $H(s)$:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s^n}{a_0 + a_1 s^n + s^{n+1}} \tag{32}$$

where the fractional order n is limited to one decade.

Its state-space model is represented on figure 9:

For this example, the following parameters have been chosen:

$a_0 = 1, a_1 = 1, b_0 = 1, b_1 = 1,$
$n = 0.4.$

Two simulations are performed. The first one (solid line) is done with a fractional integrator limited to one decade with $N = 10$ cells, $\omega_b = 0.1\, rd/s$ and $\omega_h = 1\, rd/s$. The second simulation (dotted line) is performed with an ideal fractional operator with the parameters $N = 30$ cells, $\omega_b = 10^{-5}\, rd/s$ and $\omega_h = 10^5\, rd/s$.

Figure 10 exhibits the step responses of $H(s)$ in the two cases.

This last example demonstrates clearly that the proposed fractional integrator operator is able to face conventional fractional modelling and simulation as well as non conventional ones, like fractional derivation with limited spectral range. It is also evident that neither direct numerical approximation nor diffusive representation can take into account this non conventional problem.

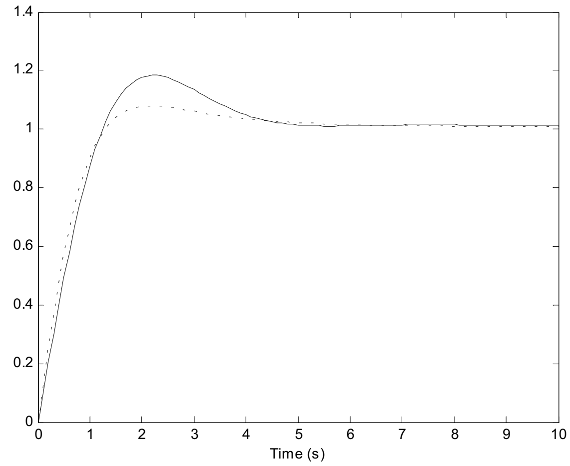

Figure 10. STEP RESPONSES OF FRACTIONAL SYSTEM WITH BOUNDED SPECTRAL RANGE (SOLID LINE $N = 10$ CELLS, $\omega_b = 0.1\, rd/s$ AND $\omega_h = 1\, rd/s$, DOTTED LINE $N = 30$ CELLS, $\omega_b = 10^{-5}\, rd/s$ AND $\omega_h = 10^5\, rd/s$

CONCLUSION

An original method for modelling and simulation of fractional systems has been presented in this paper. This modelling is based on a new fractional integrator operator, associated to a N dimensional state-space representation. Only a few parameters used for the design of the non integer action and its spectral range are necessary to characterize this operator.

Theoretical and numerical comparisons with other techniques commonly used for the simulation of fractional systems have exhibited the performances of this original approach. Its main interest is to propose a general framework for the modelling of fractional systems based on a macro state-space representation, where conventional integration is replaced by a fractional one with the help of the integrator operator. Another important feature of this new approach is its flexibility because it applies to different types of fractional models and to non conventional non integer derivation with limited spectral range.

This fractional operator has been already applied to identification problems with output error technique. Various experiments [4,5,8] have confirmed the interest and the validity of this new approach for modelling and simulation of fractional systems in diffusive applications.

REFERENCES

[1] M. Ichise, Y. Nagayanagi, and T. Kojima. An analog simulation of non integer order transfer functions for analysis of electrode processes. *J. Electroanal. Chem. Interfacial Electrochem*, 33:253, 1971.

[2] D. Matignon, B. d'Andra Novel, P. Depalle, and A. Oustaloup. *Viscothermal losses in wind instrument: a non integer model*, volume 2. System and Networks: mathematical theory and applications, akademie verlag edition, 1994.

[3] N. Retière and M. Ivanes. Modeling of electric machines by implicit derivative half-order systems. *IEEE Power Engineering Review*, pages 62–64, September 1998.

[4] J. Lin, T. Poinot, J.C. Trigeassou, and R. Ouvrard. Parameter estimation of fractional systems: application to the modeling of a lead-acid battery. In *SYSID 2000*, 12^{th} *IFAC Symposium on System Identification*, Santa Barbara, USA, 2000.

[5] J. Lin, T. Poinot, J.C. Trigeassou, and P. Coirault. Parameter estimation of fractional systems. application to heat transfer. In *ECC 2001, European Control Conference*, pages 2644–2649, Porto, Portugal, 2001.

[6] A. Oustaloup. *La dérivation non entière: théorie, synthèse et applications*. Paris, hermès edition, 1995.

[7] A. Oustaloup. *Systèmes asservis linéaires d'ordre fractionnaire*. Paris, masson edition, 1983.

[8] T. Poinot and J.C. Trigeassou. Parameter estimation of fractional models: application to the modeling of diffusive systems. In *Proc.* 15^{th} *IFAC World Congress*, Barcelona, Spain, 2002.

[9] D. Heleschewitz and D. Matignon. Diffusive realisations of fractional integrodifferential operators: structural analysis under approximation. In *Proc. IFAC Conference System, Structure and Control*, volume 2, pages 243–248, Nantes, France, 1998.

[10] G. Montseny. Diffusive representation of pseudo-differential time-operators. In *Proc. Fractional Differential Systems: Models, Methods and Applications*, volume 5, pages 159–175, Paris, France, 1998.

[11] K.S. Miller and B. Ross. *An introduction to the fractional calculus and fractional differential equations*, volume 2. Wiley-Interscience Publication, 1993.

[12] Y. Chen and K.L. Moore. Discretization schemes for fractional differentiators and integrators. *IEEE Transaction on Circuits an System 1: Fundamental Theory and Applications*, 7, October 2001.

[13] J. Beran. Statistical methods for data with long-range dependance. *Stat. Sci.*, 7:404–427, 1992.

[14] M. Fliess and R. Hotzel. Linear systems with a derivation of non integer order: introductory theory with an example. In *Proc. CESA'96 Symp. on Mod., Analysis and Sim.*, volume 1, pages 213–217, France, 1996.

[15] H.F. Raynaud and A. Zergainoh. State-space representation of fractional linear filters. In *Proc. IFAC Conference System, Structure and Control*, Nantes, France, 1998.