Stability Analysis of Fractional Adaptive High-Gain Controllers for a Class of Linear Systems General case

Samir Ladaci[1,2], Jean Jacques Loiseau[1] and Abdelfatah Charef[2]

[1] IRCCyN UMR CNRS 6597
Ecole Centrale de Nantes
1, rue de la Noë, BP: 92101 Nantes, 44321, FRANCE.
{Samir.Ladaci,Jean-Jacques.Loiseau}@irccyn.ec-nantes.fr
http://www.irccyn.ec-nantes.fr

[2] University of MENTOURI
Signal Processing Laboratory, Department of Electronics
Route de Ain Elbey, Constantine 25000, ALGERIA
afcharef@yahoo.com
http://www.umc.edu.dz

Abstract— **In this paper we demonstrate that a fractional adaptive controller based on the high gain output feedback stabilizes the class of linear, time-invariant, minimum phase, siso systems of relative degree one. We generalize the stability theorem of integer controllers to fractional order controllers whose derivative order is a real number between one and two, and introduce a new tuning parameter for the performance behaviour of the controlled plant. A simulation example is given to illustrate the effectiveness of the proposed algorithm.**

I. INTRODUCTION

Stability analysis and stability proof of fractional order control systems [5], [18] are still considered yet as open problems. This is due to the fact that the existing theory developped so far for stability proof mainly exists for integer order systems, and generaly is not applicable to fractional order control systems. Many research works have tried to give answers to this question, using numerical or analytical arguments [2], [6], [13], [14], [15], [17]. It is particularly the case for fractional adaptive control schemes [20]. The application of the theory of fractional calculus in adaptive control is just starting, but there are more and more works on this subject [10], [11], [19]. Recently, the authors have presented a new scheme of fractional order adaptive PI$^\lambda$D$^\mu$ Controller [12], based on classical integer order algorithms [4]. We showed by simulations that the use of fractional order operators improves consistently the behaviour of the controlled plants, but the weakness of such work was the lack of theoritical arguments for garanteeing the stability of such particular control schemes. All these works are based on the high gain output feedback control theory [1], [3], which is very attractive because it is not based on system identification or plant parameter estimation algorithms or injection of probing signals. The main contribution of this paper is performing analytical proof of stability of the generalization of this method to fractional order schemes for the class of linear minimum phase systems of relative degree one.

This paper is organized as follows:
In section II mathematical definitions of fractional order derivative and integral are given, section III presents the problem of fractional adaptive high gain controller stability. A theorem of stability is then proposed in section IV and its demonstration is performed in section V. A simulation example illustrates this control method in section VI. Section VII presents the conclusion of the paper.

II. MATHEMATICAL ASPECTS

The mathematical definition of fractional derivatives and integrals has been subject of several different approaches [16]. In this paper we consider the Riemann-Liouville definition, in which the fractional order integrals are defined as

$$D_a^{-\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t-\xi)^{\mu-1} f(\xi) d(\xi) \qquad (1)$$

while the definition of fractional order derivatives is

$$\begin{aligned} D_a^{\mu} f(t) &= \frac{d}{dt}\left[D_a^{-(1-\mu)} f(t)\right] \qquad (2)\\ &= \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_a^t (t-\xi)^{-\mu} f(\xi) d(\xi) \end{aligned}$$

where

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy \qquad (3)$$

is the Gamma function, $(a,t) \in \Re^2$ with $a < t$ and μ ($0 < \mu < 1$, $\mu \in \Re$) is the order of the operation.
For simplicity we will note $D^\mu f(t)$ for $D_0^\mu f(t)$.

III. PROBLEM STATEMENT

Consider an uncertain siso system described by

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (4a)$$

$$y(t) = Cx(t) \qquad (4b)$$

1-4244-0136-4/06/$20.00 ©2006 IEEE

where $t \in \Re$ is the time variable, $x(t) \in \Re^n$ is the state with n unknown, $u(t) \in \Re$ is the control and $y(t) \in \Re$ is the measured output; A, B, C, are unknown matrices of appropriate dimensions. We assume the following.

Assumption A 1: (A, B) is controllable and (C, A) is observable.

Suppose that (4a) is subject to a linear output feedback controller with gain $-k \in \Re$, i.e.,

$$u = -ky \tag{5}$$

Then the resulting feedback-controlled system can be described by

$$\dot{x} = \tilde{A}(k)x \tag{6}$$

where

$$\tilde{A}(k) \triangleq A - kBC \tag{7}$$

System (6) is asymptotically stable iff there exists $\epsilon > 0$ such that

$$Re(\lambda) \leq -\epsilon \quad \forall \lambda \in \sigma[\tilde{A}(k)]$$

where $\sigma[\tilde{A}(k)]$ denotes the set of eigenvalues of $\tilde{A}(k)$. We introduce then the following definitions.

Definition 1: System (4) is uniformly stabilizable via high gain output feedback iff there exists $\epsilon > 0$ and $\underline{k} \in \Re$ such that for all $k \geq \underline{k}$

$$Re(\lambda) \leq -\epsilon \;\; \forall \;\; \lambda \in \sigma[\tilde{A}(k)]$$

Definition 2: System (4) is minimum phase if:

$$det\begin{pmatrix} sI - A & B \\ C & 0 \end{pmatrix} \text{ has all its zeros in } C^-.$$

Definition 3: System (4) has relative degree one if:

$$CB \neq 0$$

CB is then the so-called high frequency gain.

We can add the following Theorem

Theorem 1: Consider any siso system described by (4), if in addition it is minimum-phase with relative degree one and has positive high frequency gain, then it is uniformly stabilizable by the high gain adaptive controller (6).

Proof of Theorem 1.
For the proof see for instance [3], [7], [8], [21].

□

The following assumption is made.

Assumption A 2: System (4) is minimum phase, has relative degree one and positive high frequency gain.

The adaptive controllers considered here are simply adaptive parameter versions of (5), they are given by

$$u = -\hat{k}y \tag{8a}$$

$$D^\alpha(\hat{k}) = \gamma y^2 \tag{8b}$$

where $\hat{k}(t) \in \Re$ is the adaptive parameter, $\alpha \in \Re$ with

$$1 < \alpha < 2 \tag{9}$$

and γ is any real number which satisfies

$$\gamma > 0 \tag{10}$$

IV. MAIN RESULT

Consider any system described by (4) and subject to an adaptive controller given by (8). The resulting controlled system can be described by

$$\dot{x} = \tilde{A}(\hat{k})x \tag{11a}$$

$$D^\alpha(\hat{k}) = \gamma(Cx)^2 \tag{11b}$$

where $\tilde{A}(\hat{k})$ is defined by (7). This is a system with state $(x, \hat{k}) \in \Re^n \times \Re$. The main purpose of this paper is to present the stability properties of (11). These properties are stated in the following theorem.

Theorem 2: Consider any system described by (4), satisfying Assumptions $A1$, $A2$ and subject to control given by (8). The resulting controlled system, (11) has the following properties. For each $(t_0, x_0, \hat{k}_0) \in \Re \times \Re^n \times \Re$, there exists a solution $(x(.), \hat{k}(.)) : [t_0, t_1) \to \Re^n \times \Re$, $t_0 < t_1$, of (11) with $x(t_0) = x_0$, $\hat{k}(t_0) = \hat{k}_0$. Every solution can be continued over $[t_0, \infty)$ and satisfies

$$\lim_{t\to\infty} x(t) = 0$$

$$\hat{k}(.) \text{ is bounded.}$$

V. PROOF OF THE MAIN RESULT

We shall require the following Theorem (see [3]).

Theorem 3: Consider any siso system described by

$$\dot{x} = [A - k(t)BC]x \tag{12a}$$

$$y = Cx \tag{12b}$$

where A, B, C satisfy assumptions $A1$, $A2$ and $k(.) : [t_0, t_1) \to \Re, t_0 < t_1$, is a differentiable function satisfying

$$\dot{k}(t) \geq 0.$$

There then exists $\underline{k} \in \Re$, $\epsilon > 0$ and $M > 0$ such that if $k(t) \geq \underline{k}$ for all $t \geq t_0$ and all $x(.) : [t_0, t_1) \to \Re^n$, solution of (12a), then

$$|y(t)| \leq Me^{-\epsilon(t-t_0)} \qquad \forall \; t \in [t_0, t_1)$$

Proof of Theorem 3.
Consider any $(t_0, x_0, \hat{k}_0) \in \Re \times \Re^n \times \Re$. Since the right-hand sides of (11) are continuous functions of $(x, \hat{k})$, there exists a solution $(x(.), \hat{k}(.)) : [t_0, t_1) \to \Re^n \times \Re$, $t_1 > t_0$, of (11) with $x(t_0) = x_0$, $\hat{k}(t_0) = \hat{k}_0$.
Since the right-hand side of (11a) is Lipschitz in x, it

can readily be seen that if there exists $t^* \in \Re$ such that a solution of (11a) cannot be extended past t^* then $\lim_{t \to t^*} \hat{k}(t) = \infty$.

□

We shall also need the next lemma

Lemma 1: The controller parameter function $\hat{k}$ defined in (8) satisfies

$$\dot{\hat{k}}(t) \geq 0 \tag{13}$$

Proof of Lemma 1.
Let us define $\beta \in \Re$ such that

$$\beta = \alpha - 1$$

From (9), $0 < \beta < 1$. From (8b) we have

$$\begin{aligned} D^{\alpha}(\hat{k}) &= \gamma y^2 \\ D^{1+\beta}(\hat{k}) &= \gamma y^2 \\ \dot{\hat{k}} &= D^{-\beta}(\gamma y^2) \end{aligned}$$

and from the Riemann-Liouvillle fractional order integral definition (1) we get

$$\dot{\hat{k}} = \frac{\gamma}{\Gamma(\beta)} \int_0^t (t-\xi)^{\beta-1} y^2(\xi) d(\xi) \tag{14}$$

It is obvious that the right hand side of equation (14) is positive ($\Gamma(\beta) \geq 1$ for $0 < \beta < 1$). The result follows.

□

Now we demonstrate that $\hat{k}(.)$ is bounded on every interval (finite and infinite). Suppose by contradiction that there exists $t^* \in (t_0, \infty]$ such that

$$\lim_{t \to t^*} \hat{k}(t) = \infty \tag{15}$$

The evolution of $x(.)$ satisfies

$$\dot{x} = [A - \hat{k}(t)BC]x$$

where from (13),

$$\dot{\hat{k}}(t) \geq 0$$

Let $\underline{k}$ be as defined in Theorem 3. Then, there exists $t_2 \in [t_0, t^*)$ such that $\hat{k}(t) \geq \underline{k}$ for all $t \in [t_2, t^*)$. It then follows from Theorem 3 that there exists $\epsilon > 0$ and $M \in \Re$ such that

$$|y(t)| \leq M e^{-\epsilon(t-t_2)} \qquad \forall\ t \in [t_2, t^*) \tag{16}$$

But (11a) and (16) imply that $\hat{k}(.)$ is bounded on $[t_2, t^*)$. This contradicts the original supposition, (15); hence $\hat{k}(.)$ is bounded.
The boundedness of $\hat{k}(.)$ insures that every solution of (11) can be extended indefinitely; it also implies that $y(.) \in L^2$, i.e., $\int_0^\infty y(t)^2 dt < \infty$; recall (11b).
To prove that $x(.)$ converges to zero, we rewrite (11a) as

$$\dot{x} = \overline{A}x + B\mu(t)$$

where

$$\overline{A} \triangleq A - \underline{k}BC$$

is asymptotically stable and

$$\mu(t) \triangleq (\hat{k} - \underline{k})y(t).$$

Since $y(.) \in L^2$ and $\hat{k}(.)$ is bounded, $\mu \in L^2$. Since $x(.)$ is now the state of an asymtotically stable linear system subject to an L^2 input, $x(.) \in L^2$.
Hence $\lim_{t \to \infty} x(t) = 0$.

□

REMARKS

- It can be proved that adaptive stability of the given class of nonlinear control system is possible by using the controller

$$u = -\hat{k}y \tag{17a}$$

$$D^{\alpha}(\hat{k}) = \gamma \|y\|^p \tag{17b}$$

where $p \geq 1$.
- The fractional order α of the derivative in (8b) can be considered as a supplementary tuning parameter, that can be used to improve the behaviour of the closed loop control system.
- A supplementary benefit of introducing fractional order operators is to increase the robustness of the control system against noises and perturbations as shown in several past works (see for instance [11]).

VI. SIMULATION EXAMPLE

To study the performances of this new algorithm let us apply it to a nonminimum phase continuous-time unstable system of relatif degree one given by the following transfer function:

$$G(s) = \frac{10\,(s+1)}{(s-2)(s-3)} \tag{18}$$

By introducing the adaptive controller of equations (8) with integer order derivative ($\alpha = 1$) and initial conditions $y(0) = 5$ and $\gamma = 0.3$, we obtain the result shown in Fig. 1 for the output signal and Fig. 2 for the control signal.
For the fractional order case, taking the controller descibed in (8) with fractional order derivative ($\alpha = 1.45$) and initial conditions $y(0) = 5$ and $\gamma = 0.0000001$, the output response is given in Fig. 3 and the control signal in Fig. 4.
It is obvious on the simulation results that dynamical behaviour of the fractional algorithm is better than the one of integer order particularly concerning the convergence time and the smoothness of the output signal. We remark also that the value of the parmeter gain γ is much less in the case of fractional derivative.

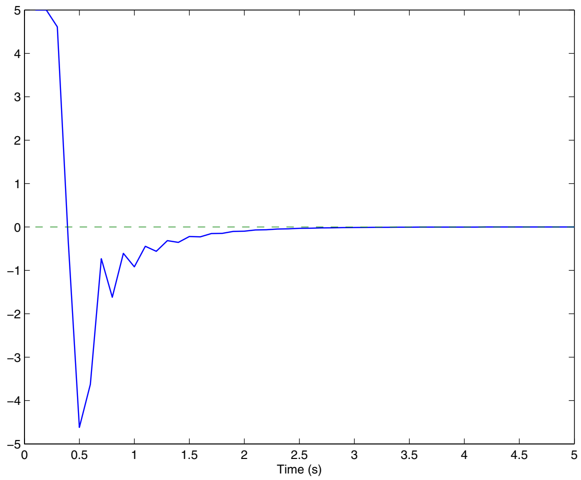

Fig. 1. Output of the integer order adaptive high gain control algorithm ($\alpha = 1$)

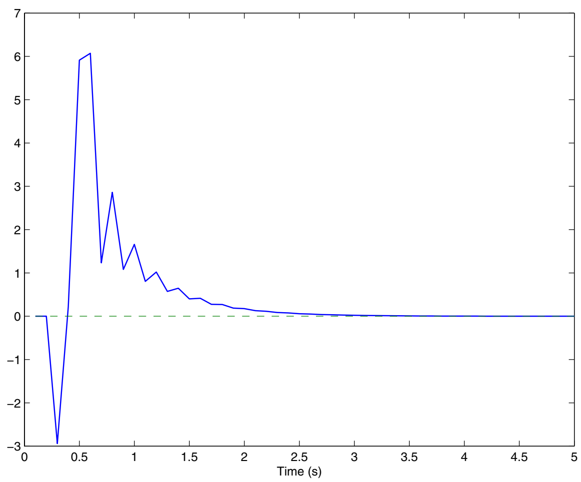

Fig. 2. Control signal of the integer order adaptive high gain control algorithm ($\alpha = 1$)

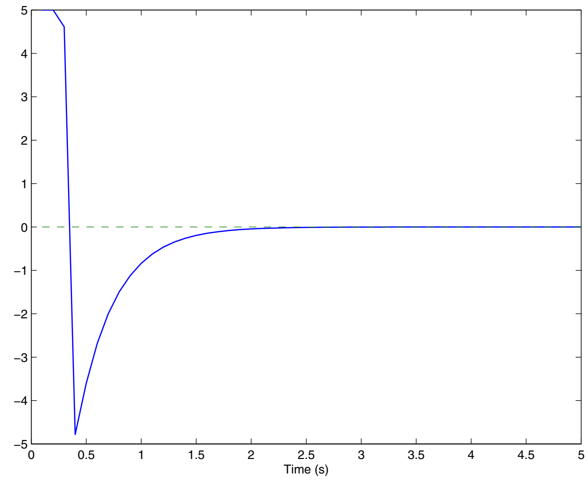

Fig. 3. Output of the fractional order adaptive high gain control algorithm with $\alpha = 1.45$

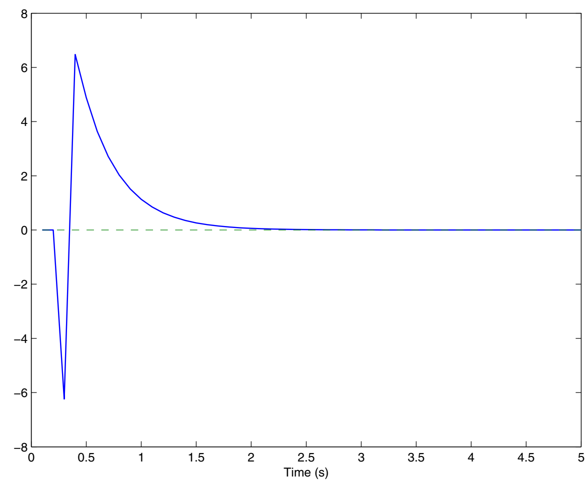

Fig. 4. Control signal of the fractional order adaptive high gain control algorithm with $\alpha = 1.45$

VII. CONCLUSION

We presented a fractional high-gain adaptive controller for a class of linear systems, allowing new tuning parameters for the improvement of closed loop performances. Stability analysis was performed and a simulation example was given to illustrate the performance improvement that can result from the introduction of fractional order operators in such control algorithms.More general fractional high gain schemes should be considered in future works, especially for wider range of values of the controller fractional derivative order. This simple adaptive control algorithm gives new arguments for the development of fractional adaptive PI^{λ} and $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controllers.

Acknowledgments

The authors express their sincere thanks for M. Malabre for his insightful comments and helpful discussions.

REFERENCES

[1] F. Allgöwer, J. Ashman and A. Ilchmann, "High-gain Adaptive λ-tracking for Nonlinear Systems," *Automatica*, vol. 33, no. 5, pp. 881–888, 1997.

[2] Y. Q. Chen and K. L. Moore, "Analytical stability bound for a class of delayed fractional-order dynamic systems," *in Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, Florida USA, December 2001.

[3] M. Corless, "First order adaptive controllers for systems which are stabilizable via high gain output feedback," *Analysis and Control of Nonlinear Systems*, Elsevier Science Publishers B. V. (North-Holland), pp. 13–16, 1988.

[4] J. C. Fan and T. Kobayashi, "A Simple Adaptive PI Controller for Linear Systems with Constant Disturbances," *IEEE Trans. on Automatic Control*, vol. 43, no. 5, pp. 733–736, 1998.

[5] T. T. Hartley and C.F. Lorenzo, "Dynamics and Control of Initialized Fractional Order Systems," *Nonlinear Dynamics*, vol. 29, pp. 201–233, 2002.

[6] C. Hwang and Y. C. Cheng, "A numerical algorithm for stability testing of fractional delay systems," *automatica*, vol. 42, pp. 825–831, 2006.

[7] A. Ilchmann, "Non-Identifier-Based High-gain Adaptive Control," *Lecture Notes in Control and Information Sciences*, no. 189, Berlin, Germany: Springer-Verlag, 1993.

[8] A. Ilchmann, "Universal Adaptive Stabilization of Nonlinear Systems," *Dynamics and Control*, vol. 7, pp. 199–213, 1997.

[9] A. Ilchmann and E. P. Ryan, "On tracking and disturbance rejec-

tion by adaptive control, " *Systems and Control Letters*, vol. 52, pp. 137–147, 2004.

[10] S. Ladaci and A. Charef, "MIT Adaptive Rule with Fractional Integration, " *in Proceedings CESA'2003 IMACS Multiconference Computational Engineering in Systems Applications*, Lille, France, July 9–11, 2003.

[11] S. Ladaci and A. Charef, "On Fractional Adaptive Control," *Nonlinear Dynamics*, vol. 43, no. 4, pp. 365–378, March 2006.

[12] S. Ladaci and A. Charef, "An Adaptive Fractional $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ Controller, " *in Proceedings TMCE 2006 International Symposium series on Tools and Methods of Competitive Engineering*, Ljubljana, Slovenia, April 18–22, pp. 1533-1540, 2006.

[13] M. P. Lazarević, "Finite time stability analysis of PD^{α} fractional control of robotic time-delay systems, " *Mechanics Research Communications*, vol. 33, pp 269–279, 2006.

[14] D. Matignon, "Stability result on fractional differential equations with applications to control processing, " *in IMACS-SMC Proceeding*, Lille, France, pp. 963–968, 1996.

[15] S. Momani and S. B. Hadid, "Lyapounov stability solutions of fractional integrodifferential equations," *International Journal of Mathematics and Mathematical sciences*, vol. 47, pp 2503–2507, 2004.

[16] K. B. Oldham and J. Spanier, *The Fractional Calculus,* Academic Press, New York, 1974.

[17] I. Petráš, Y. Q. Chen and B. M. Vinagre, "A robust stability test procedure for a class of uncertain LTI fractional order systems," *in Proceedings of the International Carpathian Conference ICCC'2002*, Malenovice, Czech Republic, pp 247–251, May 27-30, 2002.

[18] H. M. Srivastava. and R. K. Saxena, "Operators of fractional integration and their applications," *Applied Mathematics and Computation*, vol. 118, pp. 1–52, 2001.

[19] B. M. Vinagre, I. Petras, I. Podlubny and Y. Q. Chen, "Using Fractional Order Adjustment Rules and Fractional Order Reference Models in Model-Reference Adaptive Control," *Nonlinear Dynamics*, vol. 29, pp. 269–279, 2002,.

[20] B. M. Vinagre, I. Petras, I. Podlubny and Y. Q. Chen, " Stability of fractional-order model reference adaptive control," *in Proceedings of the MTNS'2002 (Open problems book)*, Notre Dame, USA, pp. 118–121, August 12-16, 2002.

[21] J. C. Willems and C. I. Byrnes, "Global adaptive stabilization in the absence of information on the sign of the high frequency gain, " *Lecture Notes in Control and Information Sciences*, vol. 62, Berlin, Germany, Springer-Verlag, pp. 49-57, 1984.