# Solution of Linear Time Invariant Differential Equations with 'Proper' Primitives

Peter W. Krempl
AVL List GmbH
Hans-List Platz 1
8020 Graz
AUSTRIA
*peter.krempl@avl.com*

*Abstract* – **The concept of 'proper' primitives of generalised complex derivatives will be presented. It will be shown, that such 'proper' primitives can be generated by a functional transformation. Within this framework of proper primitives, linear differential equations containing derivatives of arbitrary order can be solved for any set of *n* initial conditions, if the solution of the characteristic equation consists of *n* roots. In difference to the classical case, where a differential equation of the order *n* has to satisfy *n* initial conditions for all the derivatives of order 0 to *n*–1, the choice of the orders of the derivatives subject to initial conditions is arbitrary for proper primitives. This allows to take such initial conditions which are imposed by the context of the given problem. The application of this concept will be demonstrated on simple systems.**

## I. INTRODUCTION

In conventional analysis, there is an inherent difference between differential and indefinite integral operators. The classical derivative is a degenerated linear operator, since an infinite number of functions are mapped to the same function as their derivative. This causes, that the derivative of a function is unique, but their inverse operation, the indefinite integral is not, due to the arbitrary constant of integration. This results a one-dimensional infinite manifold as anti-derivatives or primitives of first order for a given function. Each consecutive step of integer integration increases the dimension of the existing manifolds of anti-derivatives by one.

Extending our analysis to differ-integrators of non-integer order rises the question, which dimensionality has the manifold of their primitives. Since most of the definitions of fractional integrals or derivatives are based on definite integrals, involving a fixed lower or upper boundary, this problem doesn't arise in this case. On the other side, their results differ due to the different integration limits assumed in the definition, yielding the paradox situation, that the integrals are unique, but the derivatives depend on an integration boundary. In some applications it is very strange or haphazard to select an appropriate integration boundary for the derivative, which is unnecessary for classical derivatives of integer order. In mechanics for instance, the acceleration is the first derivative of the velocity with respect to the time and independent of the chosen origin of time. This shows that classical derivatives describe instantaneous or local relations. A nice example of fractional derivatives in physics for which no integration boundary is meaningful arises in the fractional description of the fractional wave function describing spin ±1/2 of the fundamental particles in physics [1].

### A. Derivatives of a Constant

If we want to have perfect symmetry between derivatives and integrals, the latter expressed as derivatives of negative order, we are forced to ask for a definition of primitive functions which are unique for a given function. such a concept of unique primitives can be simply introduced by neglect of the constants of integration. But doing this, we have still the fact, that different functions have the same derivatives. The reason is the vanishing of the derivative of a constant in classical analysis. In fractional calculus, the derivatives of a constant depend on the definition. In the most frequently used Riemann definition, these derivatives become:

$$\frac{d^{\alpha}C}{dx^{\alpha}} = \frac{C}{\Gamma(1-\alpha)} x^{-\alpha} \tag{1}$$

whereas according to Caputo's and other definitions the fractional derivatives of a constant vanish [2].

### B. 'Proper' Primitives in Conventional Analysis

We can define 'proper' primitives which are unique functions of their derivatives already in classical analysis. Let us denote the result of the $k$-times consecutive indefinite integration of the function $f(z)$ with respect to $z$ as $I_z^k\left[f(z)\right]$. We have then:

$$I_z^0[f(z)] \equiv f(z) \tag{2}$$

with the *recursion formula:*

$$I_z^{k+1}[f(z)] \equiv \int I_z^k[f(z)]\,dz \tag{3}$$

In classical calculus, only $I_z^0\left[f(z)\right]$ is unique for a given $f(z)$. According to the fundamental theorem of calculus, $I_z^1\left[f(z)\right]$ can be any function $F_C(z)$ satisfying:

$$\frac{d}{dz} F_C(z) = f(z) \tag{4}$$

and they can differ only by an additive constant of integration $C$, which has to be added since the derivative of this constant vanishes. These functions are generally denominated as the primitive functions or primitives of $f(z)$. If we omit these additive constants, like it is done in many tables of indefinite integrals, they all become identical. Let us denote this unique function with $F(z)$ and call it the 'proper' primitive of 1[st] order of $f(z)$. The 'proper' primitive of 2[nd] order is obtained by indefinite integration of $F(z)$, again

1-4244-0136-4/06/$20.00  

omitting the integration constant. Following this concept to higher orders, we obtain a unique set of 'proper' functions as 'proper' integrals of higher, but integer order of the function $f(z)$, like we have a unique set of classical derivatives of integer order.

According to this definition, the 'proper' primitives are those primitives, which we would obtain, if the derivative of a constant wouldn't vanish. That's the reason, why these primitives can be denominated as 'proper'. It has the big advantage, that these primitives are identical with the usually tabulated primitive functions.

## II. 'PROPER' PRIMITIVES

In fractional calculus, or more general in generalised complex calculus, we should not need to distinguish between derivatives and integrals, because they differ only by the sign of the order $\alpha$. To take this fact into account we are forced to introduce the concept of unique 'proper' primitives.

### *A. Definition of 'Proper' Primitives, Rules of Coincidence, and Recursion*

Let us make the following definition of 'proper' primitives without restrictions to derivatives of real order:

*Definition*: The *proper primitives* of a function $f(z)$ are the functions $P(\alpha; f, z)$ with the following properties:

$$P(f,z;\alpha) \equiv f^{(\alpha)}(z) \qquad \alpha \in \mathbf{C} \tag{5}$$

where $f^{(\alpha)}(z)$ denotes the generalised complex derivative of order $\alpha$ of $f(z)$ with respect to $z\in\mathbf{C}$, with. the special case:

$$P(f,z;0) = f(z) \tag{6}$$

This means that, in the general case of complex functions and derivatives, we have uniquely defined functions $P(f, z; \alpha) = f^{(\alpha)}(z)$ over $\mathbf{C}^2$. If we restrict our considerations to derivatives of arbitrary real order $\alpha$ with respect to real arguments $x$, $P(f, x; \alpha)$ becomes a function over the $(\alpha,x)$-plane.

These derivatives have to coincide with the classical derivatives of integer order if this order approaches a natural number $n$, which means:

*I. Rule of Coincidence*:

$$\lim_{\alpha\to n} f^{(\alpha)}(z) = \frac{d^n}{dz^n} f(z) = P(f,z;n) \quad n\in\mathbf{N}_0 \tag{7}$$

The classical first order derivative of a generalised derivative of order $\alpha$ should become equal to the generalised derivative of the order $\alpha+1$, i.e.:

*II. Rule of Recursion*:

$$\frac{d}{dz} P(f,z;\alpha) = P(f,z;\alpha+1) \tag{8}$$

This can be used to calculate all the generalised derivatives of real order of $f(z)$ if they are known for $0<\alpha<1$.

### *B. Group Properties*

Generalisation of rule II for all derivatives asks for the validity of the associative law for the generalised derivatives, yielding the following rule:

*III. Rule of Composition*:

$$\frac{d^\beta}{dz^\beta} P(f,z;\alpha) = P(f,z;\alpha+\beta) \tag{9}$$

This notation will only be correct, if we assume the validity of the following commutative law:

*IV. Rule of Commutation*:

$$\frac{d^\beta}{dz^\beta} P(f,z;\alpha) = \frac{d^\alpha}{dz^\alpha} P(f,z;\beta) \tag{10}$$

If these rules are satisfied, the proper primitives forms an Abelian group with respect to the order of derivation. In fractional calculus it is shown that, if negative orders (integrals) are involved, the commutative law doesn't apply for derivatives of all orders for most of the definitions of fractional differentiation [2-6] .

### *C. Continuity of the Proper Primitives with Respect to their Order*

Well behaved systems are characterised by the property that minor changes of the system parameters will cause only minor changes of the system behaviour. In consequence, if the order of the derivatives become an adjustable parameter of the system, then minor variation of the order of derivation should cause only minor changes of the system functions. This means that we are asking for a new type of continuity and define:

> *Definition*: A function $f(z)$ which is analytic over a domain $z\in \mathrm{D}$ is continuous with respect to the order $\alpha$ of its derivative $f^{(\alpha)}(z)$ over the domain $\mathrm{M}$ ($\alpha\in\mathrm{M}\subset\mathbf{C}$), if and only if for every $z\in\mathrm{D}$, and for every $\alpha\in\mathrm{M}$, and every real number $\delta>0$, there exists a real number $\varepsilon>0$ such that whenever $|\alpha-\alpha_0|<\delta$ with $\alpha_0\in\mathrm{M}$ the relation:
>
> $$|P(f, z;\alpha)-P(f, z;\alpha_0)|<\varepsilon \tag{11}$$
>
> is satisfied.

If $P(f, z; \alpha)$ is a meromorphic function of $\alpha$, it will be continuous with respect to the order $\alpha$. The beauty of fractional calculus lies in the fact, that for any given function $f(z)$ we can regard its derivatives $f^{(\alpha)}(z)$ as a function $P(f, z; \alpha) = f^{(\alpha)}(z)$ of $\alpha$, $f$, and $z$.

Obviously, this continuity cannot be asked for classical derivatives of only integer order, but in the generalised case it makes sense.

*V. Rule of Continuity*:

Proper primitives of regular functions $f(z)$ of arbitrary order $\alpha$ should be continuous with respect to their order $\alpha$ over the entire domain $D$, where $f(z)$ is analytic.

It might be noticed, that the proper primitive $P(f, z; \alpha)$ include the function $f(z)$ and all its complex derivatives (or integrals) of order $\alpha$. If $P(f, z; \alpha)$ is an analytic function of the complex variable $\alpha$ and defined for $\alpha\in\mathbf{R}$, than its analytic continuation is uniquely determined over $\mathbf{C}$. From:

$$P(f^{(\beta)}, z; \alpha) = P(f, z; \alpha+\beta) \tag{12}$$

follows, that the proper primitives represent a complete family of functions with their generalised complex derivatives (or integrals) and their complex multiples, forming the set $\{f^{(\alpha)}(z)\}\equiv\{\lambda P(f, z; \alpha)\}$ with arbitrary $\lambda\in\mathbf{C}\backslash 0$ and $\alpha\in\mathbf{C}$.

*Corollary*: If $\{P(f, z; \alpha)\}\cap\{P(g, z; \beta)\}\neq\{0\}$, than $\{P(f, z; \alpha)\}=\{P(g, z; \beta)\}$ or: If a function $g(z)\in\{f^{(\alpha)}(z)\}$, than there exist complex numbers $\gamma,\lambda\in\mathbf{C}$, such that:

$$g(z) = \lambda P(f, z; \gamma). \tag{13}$$

*D. Differentiability with Respect to the Order*

Perhaps one of the most important aspects of generalised calculus lies in the fact, that the order $\alpha$ of differentiation can be continuously varied, like the argument $z$. If the proper primitives $P(f, z; \alpha)$ are differentiable with respect to $\alpha$ as variable, we can introduce also generalised derivatives with respect to the order $\alpha$. This raises the question, if the generalised derivatives of $P(f, z; \alpha)$ with respect to the variables $\alpha$ and $z$ can be interchanged, i.e. if they satisfy the relation:

$$\begin{aligned}&\frac{\partial^\beta\partial^\lambda P(f,z;\alpha)}{\partial z^\beta\partial\alpha^\lambda} = \\ &= \frac{\partial^\lambda\partial^\beta P(f,z;\alpha)}{\partial\alpha^\lambda\partial z^\beta} = \frac{\partial^\lambda P(f,z;\alpha+\beta)}{\partial\alpha^\lambda}\end{aligned} \tag{14}$$

It might be mentioned that the extension of the order of derivatives to complex numbers allows us to apply function theoretical results to generalised derivatives also of real order. This mathematical application is perhaps more important than technical applications of derivatives with partly imaginary order.

## III. GENERALISED COMPLEX DERIVATIVES

Now, let us look for functions which satisfy the above definition and rules for 'proper' primitives. To avoid fractional derivatives arising from fractional integrals with fixed lower or upper integration limits, we have to make a new approach towards fractional calculus. In this chapter, we introduce generalised complex derivatives (GCDs) in an axiomatic way. The denomination as GCD was chosen to distinguish these new 'fractional' derivatives, which are not restricted to real order, from the well known derivatives like the Cauchy, Grünwald-Letnikov, or the most frequently used Riemann-Liouville ones.

*A. K-Transformation*

The following approach to generalised derivatives yielding 'proper' primitives satisfying the rules I to V uses a transformation (in the following denoted as K-transform) of the space $\boldsymbol{f}(D)$ of complex functions $f(z)$ which are absolutely continuous and differentiable over a domain $D\subset\mathbf{C}$ of the complex plane into the abstract space $\boldsymbol{F}\subset\mathbf{C}$ of their K-transforms $F(\nu) = \mathsf{K}[f(z), z, \nu]$, which are defined by the following rules:

A1: *Definition*: The K-transform $F(\nu)$ of a function $f(z)\in\boldsymbol{f}(D)\subset\mathbf{C}$ is the function $F(\nu)\in\boldsymbol{F}\subset\mathbf{C}$ with the following property:

$F(\nu)=\mathsf{K}[f(z), z, \nu]$ is the K-transform of $f(z)$ then and only then if $F(\nu+\alpha)$ is the K-transform of its *generalised complex derivative (GCD)* $f^{(\alpha)}(z)$ to the complex order $\alpha$ (15)

It might be noticed, that this definition defines the K-transformation as well as the associated GCDs. In the K-space, the K-transforms of the generalised derivative of a given function $f(z)$ is represented by the function $F(\nu+\alpha)$ obtained by adding the order $\alpha$ of the derivative to the argument $\nu$. Thus, in this space, derivation is reduced to a simple shifting of the argument $\nu$. This must also be valid for integer values of $\alpha$. This means, that if $f(z)$ transforms to $F(\nu)$, its derivative $df(z)/dz$ transforms to $F(\nu+1)$.

A2: *Linearity*: The K-transform is a linear homogeneous non-degenerated operator.

$$\mathsf{K}\left[\sum_{k=1}^{n} c_k f_k(z), z, \nu\right] = \sum_{k=1}^{n} c_k \mathsf{K}\left[f_k(z), z, \nu\right] \tag{16}$$

A3: *Product Rule*: If $f(z)$ and $g(z)$ are analytic functions $f(z)$ over a domain $D\subset\mathbf{C}$, and $F(\nu) = \mathsf{K}[f(z),z,\nu]$ and $G(\nu) = \mathsf{K}[g(z),z,\nu]$ than their product $f(z).g(z)$ has the K-transform:

$$\mathsf{K}\left[f(z)g(z), z, \nu\right] = (F*G)(\nu) \tag{17}$$

where $(F*G)(\nu)$ denotes the convolution:

$$\begin{aligned}(F*G)(\nu) &= \int_{-\infty}^{+\infty}\binom{\nu}{\mu}F(\mu)\,G(\nu-\mu)d\mu = \\ &= \int_{-\infty}^{+\infty}\binom{\nu}{\mu}F(\nu-\mu)\,G(\mu)d\mu\end{aligned} \tag{18}$$

A4: *Scaling Rule*: If $F(\nu) = \mathrm{K}[f(z),z,\nu]$, than

$$\mathrm{K}\left[f(az),z,\nu\right] = a^{\nu}\mathrm{K}\left[f(z),z,\nu\right] = a^{\nu}F(\nu) \quad (19)$$

As corollary of A3 we obtain the:

*Generalised Leibniz Rule*: The $\alpha$–th generalised complex derivative of the product $f(z).g(z)$ is given by:

$$(f(z)g(z))^{(\alpha)} = \mathrm{K}^{-1}\left[(F*G)(\nu+\alpha),\nu,z\right] \quad (20)$$

### B. 'Proper' Primitives as Generalised Complex Derivatives based on the K-Transformation

It was shown in [7],[8] that the following proper primitives (or generalised complex derivatives), which are particular relevant for the solution of linear time invariant differential equations can be deduced from the axioms A1 to A4:

$$P(z^p,z;\alpha) = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} z^{p-\alpha} \quad \text{with } p \notin \mathbf{Z}^- \quad (21)$$

and, assuming $p \neq 0$:

$$P(e^{pz},z;\alpha) = p^{\alpha}e^{pz} \quad (22)$$

$$P(ze^{pz},z;\alpha) = p^{\alpha-1}(pz+\alpha)e^{pz} \quad (23)$$

$$P(z^2e^{pz},z;\alpha) = p^{\alpha-2}((pz)^2 + 2\alpha pz + \alpha^2 - \alpha)e^{pz} \quad (24)$$

$$P(\sin(pz),z;\alpha) = p^{\alpha}\sin(pz+\pi\alpha/2) \quad (25)$$

$$P(\cos(pz),z;\alpha) = p^{\alpha}\cos(pz+\pi\alpha/2) \quad (26)$$

$$P(\sinh(pz),z;\alpha) = p^{\alpha}\sinh(pz+\pi\alpha/2) \quad (27)$$

$$P(\cosh(pz),z;\alpha) = p^{\alpha}\cosh(pz+\pi\alpha/2) \quad (28)$$

Within this frame of proper primitives we have the coexistence of derivatives known as Riemann derivatives (Riemann-Liouville ${}_0D_z^{\alpha}$) like (21), and derivatives arising from Liouville's definition (Riemann-Liouville ${}_{-\infty}D_z^{\alpha}$) like (22), (25) to (28), which are usually thought to mutually exclude them, and which are listed in [2]. These proper primitives are defined over the complete complex plane, except (21), which has a singularity of at $z$=0. It might be noticed that the proper primitive of a constant $C$ is included in $\{\lambda P(z^p, z; \alpha)\}$ as $C/\Gamma(1+p)P(z^p, z; p+\alpha)$ since:

$$\frac{C}{\Gamma(p+1)}P(z^p,z;p+\alpha) = CP(z^0,z;\alpha) = \frac{C}{\Gamma(1-\alpha)}z^{-\alpha} \quad (29)$$

The proper primitive $P(C, z; \alpha)$ of the constant $C$ is therefore:

$$P(C,z;\alpha) = \frac{C}{\Gamma(1-\alpha)}z^{-\alpha} \quad (30)$$

This is very important, because the existence of the proper primitives is based on the non-vanishing of the derivatives of a the constant in the general case. Of course, the derivatives $C^{(n)}$ vanish for $n \in \mathbf{N}\backslash 0$, but not in their proximity.

The rule (24) is not listed in [2], but derived in [7], and important for the following chapter. Perhaps the main advantage of these generalised complex derivatives or proper primitives are that they give us only the raisins of the cake by the consistency of the rule for the power function arising in the Riemann's case, and the rule for the exponential function arising from Liouville's case. This is no contradiction, because the Riemann derivatives corresponding to (21) are only defined by Riemann fractional integral for the positive half plane of $\mathbf{C}$. Their analytic continuation into the negative half plane gives (21). The derivatives of the exponential function in the Riemann case are gained from the derivatives of its series expansion resulting Mittag-Leffler functions. However, it was shown in [8] that Taylor expansions don't coincide with their original function as soon as non-integer derivatives are taken into account, because their derivatives differ by a series of (classically hidden) distributions from the derivatives of the expanded function.

## IV. LINEAR TIME INVARIANT DIFFERENTIAL EQUATIONS

### A. Generalised Linear Time Invariant Differential Equations

Let us now consider the homogeneous generalised differential equation:

$$\sum_{k=0}^{n} c_k f^{(\alpha_k)}(t) = 0 \quad (31)$$

where the $\alpha_k$ denote the arbitrary order of the derivatives and $c_k$ constant coefficients. We assume one of this $\alpha_k$ to be 0. Inserting

$$f^{(\alpha_k)}(t) = P(\exp(st),t;\alpha_k) = s^{\alpha_k}\exp(st) \quad (32)$$

into (31) gives us the characteristic equation:

$$\sum_{k=0}^{n} a_k s^{\alpha_k} = 0 \quad (33)$$

It was shown in [9] that each solution $s_k$ of (33) with the multiplicity $m_k$ yields the set of solutions:

$$f_{kl}(t) = c_{kl}t^l\exp(s_k t) \quad \text{with } l = 0,\ldots,m_k-1 \quad (34)$$

of (31) with arbitrary coefficients $c_{kl}$ which can be used to satisfy the given initial conditions. By this fact, we can extend some theorems for linear differential equations with constant coefficients to these linear fractional differential equations.

But, it might be noticed, that we obtained a new freedom. In classical calculus, the arbitrary $n$ coefficients $a_{kl}$ are usually determined by the initial values $f^{(k)}(0)$ of the derivatives from $k$ = 0 up to the order $k$=$n$–1. In the generalised complex calculus, we can use initial values for any set of $n$ different GCDs. This is very important in the applications. If we investigate free oscillations which are described by the generalised differential equation:

$$f^{(2)}(t) + a f^{(\alpha)}(t) + b f(t) = 0 \quad (35)$$

with $0<\alpha<2$, we can use the observable values of the function $f(t_0)$ and its derivative $f^{(1)}(t_0)$ at any time $t_0$ for the determination of the two coefficients for the desired solution. This solution is valid for all $t$, so we are able to reconstruct the time evolution with these solutions.

This is a difference to methods using Riemann-Liouville derivatives with 0 as lower integration limit. These solutions are only valid for $t \geq 0$

*B. Systems of Generalised Linear Time Invariant Differential Equations*

In the following, the concept of the application of proper primitives to such systems will be only roughly outlined. Let us now consider the following system of $n$ generalised linear time invariant differential equations:

$$\sum_{k=1}^{n} D_{ik} x_k(t) = u_i(t) \quad \text{with} \quad i = 1,...,n \quad (36)$$

where the $x_k(t)$ are different functions, the $D_{ik}$ are linear generalised complex differential operators of the form:

$$D_{ik} = a_{ikj} \sum_{j=1}^{m_{ik}} \frac{d^{\alpha_{ikj}}}{dt^{\alpha_{ikj}}} \quad (37)$$

where $\alpha_{ikj}$ are the orders of GCDs (including for simplicity at least one 0), $m_{ik}$ are the numbers of different GCDs in the operator $D_{ik}$, and $a_{ikj}$ are constant coefficients, and the functions $u_i(t)$ are arbitrary input functions.

To solve the associated homogeneous differential equations for the system, we set

$$x_k(t) = \lambda_k \exp(st) \quad (38)$$

where we chose one multiplier, say $\lambda_n=1$, and the other multipliers $\lambda_k$ are to be determined. This gives using the proper primitives (22) for the homogeneous equations the expressions:

$$\sum_{k=1}^{n} D_{ik} x_k(t) = \left( \sum_{k=1}^{n} \lambda_k \sum_{j=1}^{m_{ik}} a_{ikj} s^{\alpha_{ikj}} \right) \exp(st) = 0 \quad i = 1,...,n \quad (39)$$

yielding the $n-1$ linear equations:

$$\sum_{k=1}^{n-1} \lambda_k \sum_{j=1}^{m_{ik}} a_{ikj} s^{\alpha_{ikj}} = -\sum_{j=1}^{m_{in}} a_{inj} s^{\alpha_{inj}} \quad i=1,...,n-1 \quad (40)$$

for the $n-1$ multipliers $\lambda_k$. (40) is a linear equation for the multipliers $\lambda_k$, whose solution is straight forward. We can introduce these multipliers $\lambda_k(s)$, which in general will be functions of $s$, into the matrix elements

$$D_{ik}(s) = \lambda_k(s) \sum_{j=1}^{m_{ik}} a_{ikj} s^{\alpha_{ikj}} \quad (41)$$

which become functions of $s$ only.

The determinant of the $(n,n)$-Matrix $\mathsf{D}(s)= (D_{ik}(s))$ gives the characteristic equation:

$$\text{Det } \mathsf{D}(s) = 0 \quad (42)$$

which has to be solved for $s$. Unfortunately, there exist to the knowledge of the author no general theorem about the total number $N$ of solutions $s_r$ ($r$=1,..,$N$) for the characteristic equation (42). In simple cases, where the orders of the generalised derivatives are rational numbers, (42) can be transformed to an algebraic equation [10]. However, for each root $s_r \neq 0$, we obtain a solution of the form:

$$x_{kr}(t) = \lambda_k \exp(s_r t) \quad k=1,...,n \quad (43)$$

If $s_r \neq 0$ is a multiple root with the multiplicity $m_r>1$ we have further solutions:

$$x_{krm}(t) = \lambda_k t^m \exp(s_r t) \quad \text{with } m=1,...,m_r-1 \quad (44)$$

The general solution of the homogeneous equation is therefore:

$$x_k(t) = \sum_{r=1}^{N} \sum_{m=1}^{m_r-1} c_{krm} x_{krm}(t) \quad (45)$$

where the arbitrary coefficients $c_{krm}$ are used to match the imposed initial or other conditions, if the system is a homogeneous one. In the general case of an inhomogeneous system, we have first to find any particular solution $x_{kp}(t)$ of the inhomogeneous system, and add the general solutions (44) to get the general solution of the inhomogeneous system:

$$x_k(t) = x_{kp}(t) + \sum_{r=1}^{N} \sum_{m=1}^{m_r-1} c_{krm} x_{krm}(t) \quad (46)$$

The coefficients are again to be determined from the imposed conditions.

## V. APPLICATIONS

The following illustrative examples show the simplicity of this concept in practical applications.

*A. Generalised Relaxor*

The differential equation of generalised relaxor is given as:

$$\frac{d^\alpha}{dt^\alpha} x(t) + \tau^{-\alpha} x(t) = 0 \quad (47)$$

where the order $\alpha$ of the derivative is usually assumed to be $0<\alpha<1$.

The characteristic equation

$$s^\alpha + \tau^{-\alpha} = 0 \quad (48)$$

has the root $s = \exp(i\pi/\alpha)/\tau$ yielding the general solution:

$$x(t) = a \exp\left\{ \left[ \cos(\pi/\alpha) + i \sin(\pi/\alpha) \right] \frac{t}{\tau} \right\} \quad (49)$$

of the homogeneous equation (47). Choosing $a=x_0\ \exp(t_0/\tau)$ gives us the solution passing through $x_0$ at $t = t_0$. This system includes periodic solutions for $\alpha=2/(2n+1)$, $n\in\mathbf{N}$ and becomes very sensitive to $\alpha$ if $\alpha$ tends toward zero.

The differential equation:

$$\frac{d}{dt}x(t)+2\lambda\frac{d^{1/2}}{dt^{1/2}}x(t)+\lambda^2x(t)=0 \qquad (50)$$

has the characteristic equation:

$$s+2\lambda\sqrt{s}+\lambda^2=(\sqrt{s}+\lambda)^2=0 \qquad (51)$$

with the root $\sqrt{s} = -\lambda$ with multiplicity 2. Hence, we have the general solution:

$$x(t)=(a+bt)\exp\left[\lambda^2t\right] \qquad (52)$$

which can be easily proofed using the proper primitives (22) and (23). The semi-derivative of this function is:

$$x^{(1/2)}(t)=-\frac{(2a\lambda^2+b+2b\lambda^2t)}{2\lambda}\exp\left[\lambda^2t\right] \qquad (53)$$

Performing this calculation, we have to keep in mind that the negative branch of $\sqrt{(\lambda^2)}$ has to be taken. The solution satisfying the initial conditions $x(0)=x_0$ and $x^{(1)}(0) = 0$ is given with:

$$x(t)=x_0(1-\lambda^2t)\exp\left[\lambda^2t\right] \qquad (54)$$

and the solution satisfying the initial conditions $x(0)=0$ and $x^{(1)}(0) = v_0$ denotes:

$$x(t)=v_0t\exp\left[\lambda^2t\right] \qquad (55)$$

If $\lambda=i/\sqrt{\tau}$, we obtain the behaviour of a relaxation process, but with two degree of freedoms:

$$x(t)=(a+bt)\exp\left[-t/\tau\right] \qquad (56)$$

Finally, it is worthwhile to remember that the general solution (52) is not restricted to $t\geq 0$.

### *B. Generalised Oscillator*

Now let us look for the oscillator equation:

$$\frac{d^2}{dt^2}x(t)+\tau^\alpha\omega^2\frac{d^\alpha}{dt^\alpha}x(t)+\omega^2x(t)=0 \qquad (57)$$

which is well known to describe oscillations with a dissipative term, proportional to a fractional derivative. The characteristic equation is:

$$s^2+\tau^\alpha\omega^2s^\alpha+\omega^2=0 \qquad (58)$$

with real $0<\alpha<1$. This equation can be solved numerically and yields at least two roots, or one root with the multiplicity 2. With real parameters $\tau$ and $\omega$, the typical solutions are damped oscillations.

## VI. CONCLUSIONS

It was shown, that the concept of 'proper' primitives as the unique results of integration or differentiation yields a considerable simplification of the problem of initialisation of the solutions of linear time invariant differential equations. But this concept is not restricted to this simple type of differential equations. It can be applied to all types of generalised differential equations and different types of boundary conditions, due to the fact that the proper primitives are independent from any a priori predefinition of boundaries by the usage of generalised derivatives.

## VII. REFERENCES


[1] P. W. Krempl, Some Applications of Semi-Derivatives and Semi-Integrals in Physics. *Proc. ENOC 05*, Eindhoven, Aug. 2005, ID11-363, 10p

[2] I. Podlubny, *Fractional Differential Equations*. 340p, Academic Press, San Diego, 1999

[3] K. B. Oldham, J. Spanier, *The Fractional Calculus*. Academic Press, New York, 1974

[4] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives*. Gordon & Breach, London, 1993

[5] P.L. Butzer, U. Westphal, An introduction to fractional calculus. I, n R. Hilfer R. (Ed.), *Application of fractional calculus in physics*. World Scientific Publ. Comp., Singapore, 2000, 1-85

[6] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006

[7] P. W. Krempl, On the 'Proper' Primitives in Fractional Calculus. *Proc. 2nd IFAC Workshop on Fractional Differentiation and its Applications (FDA'06)*, Porto, July 2006 (to be published)

[8] P. W. Krempl, Generalised Fractional Differential Equations using 'Proper' Primitives. *Proc. 5th Intern Conf. on Engineering Computational Technology ECT 2006*, Las Palmas de Gran Canaria, 12-15 Sept 2006 (to be published)

[9] L. M. B. C. Campos, "On the solution of some simple fractional differential equations", *J. Math. & Math. Sci.*, **13**, 3, 1990, 481-496

[10] A. Oustaloup, "*La dérivation non entière*", Hermès, Paris, 1995, 81-85