Identification of diffusive interfaces using a simplified fractional integrator. Part 1 : linear case

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Abstract— **This paper presents a new approach to model heat transfer problems and more particularly diffusive interfaces. These interfaces can be modeled with fractional models, based on the use of a fractional integration operator. Succinctly, the approach presented in this paper consists in identification of the behavior of diffusive interfaces starting from frequency considerations derived from the analysis of a diffusion problem. Using a simplified linear fractional model, this identification is carried out by the output-error technique. A numerical simulation is presented in this paper in order to illustrate the improvements of the proposed fractional integrator in the linear case.**

I. Introduction

The modeling of diffusion interfaces is needed by a great number of applications. These processes are characterized by a fractional behavior. Concretely, such phenomenon appears in the case of an induction machine, with Foucault currents inside rotor bars [2], [8], [10], [17]. It appears also in the case of heat transfer between the flux and the temperature at the interface of the process [1], [9]. Several approaches have been developed to model this type of phenomenon. The solution proposed in this paper is based on the definition of an integrator operator [11], [12]. A first solution consists to use a fractional model like [12], [18] :

$$H_n(s) = \frac{b_0}{a_0 + s^n} \tag{1}$$

This model is based on the use of a non integer integrator $\frac{1}{s^n}$ truncated in the frequency domain. The fractional order n is fitted by identification, using time responses ; so, the frequential approximation of the diffusion interface is indirectly performed. This approximation is accurate in a frequency domain corresponding to the spectrum of the input. Practically, n is estimated in such a way that the frequency response is correctly fitted in low and medium frequencies. On the other hand, this model is not able to give satisfaction for $\omega \rightarrow \infty$ when estimated n is different of the theoretical value 0.5. Nevertheless, the interest of this model [11], [12] is its ability to approximate the dynamical behavior of diffusion interfaces using a restricted number of parameters. However, this model does not give the best approximation of the system dynamics, since $\theta \rightarrow -n \times 90°$ when $\omega \rightarrow \infty$.

In order to improve the fractional behavior of this model, and particularly its high frequency behavior (quick transients), a second approach has been used [3], [4], [5], [6] ; it consists to use a model with two fractional integrators :

$$H_{n_1,n_2}(s) = \frac{b_0 + b_1\, s^{n_1}}{a_0 + a_1\, s^{n_1} + s^{n_1+n_2}} \tag{2}$$

In imposing $n_2 = 0.5$, and then adjusting the order n_1 and the parameters b_0, b_1, a_0, and a_1, one can get a higher approximation capability, with respect of the physics (order $n_1 + n_2 - n_1 = 0.5$ at short times) and able to fit to the influence of the system geometry thanks to n_1. According to identification results this model ($H_{n_1,n_2}(s)$) has shown its interest to approximate the fractional behavior in spite of a higher complexity than the previous model ($H_n(s)$). In order to reduce this complexity, a first modification has dealt with the replacement of the two fractional integrators by a simple once, where the fractional order n_i is varying with frequency. Finally, we show in this paper that it is possible to simplify this new operator, without sacrificing its performances.

This paper begins by the definition of the fractional integrator and its application to the simulation and the identification of the non integer order model. Then, the modeling of varying fractional order integrator is presented. The model with two fractional integrators is then presented and tested in simulation. The parsimonious and simpler integrator is finally validated using a physical diffusion simulation.

II. Modeling of the non integer systems with a frequency varying order integrator

A. Fractional integrator

Let us consider the Bode diagrams of an integrator truncated in low and high frequencies (figure 1), [11], [12], [18].

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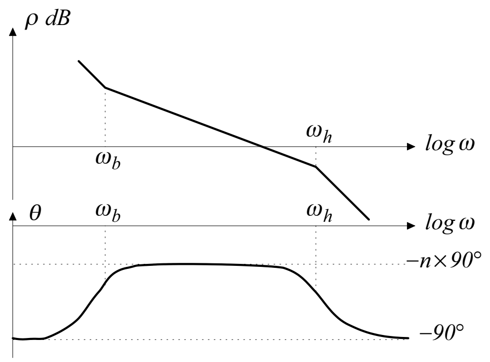

Fig. 1. Bode diagrams of the fractional integrator

It is composed of three parts. The intermediary part corresponds to non-integer action, characterized by the order n. In the two other parts, the integrator has a conventional action, characterized by its order 1.
In this way, the operator $I_n(s)$ is defined as a conventional integrator, except in a limited band $[\omega_b\,,\,\omega_h]$ where it acts like $\frac{1}{s^n}$. The operator $I_n(s)$ is defined using a fractional phase-lead filter [15] and an integrator $\frac{1}{s}$:

$$I_n\left(s\right)=\frac{G_n}{s}\prod_{i=1}^{N}\frac{1+\frac{s}{\omega'_i}}{1+\frac{s}{\omega_i}} \tag{3}$$

This operator is completely defined by :

$$\omega_i=\alpha\,\omega'_i,\qquad \omega'_{i+1}=\eta\,\omega_i,\qquad n=1-\frac{\log\alpha}{\log\alpha\,\eta}$$

where α and η are recursive parameters linked to the fractional order n.

Using (3), the corresponding state-space representation is :

$$\underline{\dot{x}}_I=A_I^*\,\underline{x}_I+\underline{B}_I^*\,u \tag{4}$$

where $A_I^*=M_I^{-1}A_I$, $\underline{B}_I^*=M_I^{-1}\underline{B}_I$, and

$$M_I=\begin{bmatrix}1&0&\cdots&0\\-\alpha&1&&\vdots\\\vdots&\ddots&\ddots&0\\0&\cdots&-\alpha&1\end{bmatrix}\qquad \underline{B}_I=\begin{bmatrix}G_n\\0\\\vdots\\0\end{bmatrix}$$

$$A_I=\begin{bmatrix}0&0&\cdots&0\\\omega_1&-\omega_1&&\vdots\\\vdots&\ddots&\ddots&0\\0&\cdots&\omega_N&-\omega_N\end{bmatrix}\qquad \underline{x}_I=\begin{bmatrix}x_1\\x_2\\\vdots\\x_{N+1}\end{bmatrix}$$

B. Fractional integrator with frequency varying order

$I_{nv}(s)$ is the new integrator designed by the association of two integrators I_{n_1} and I_{n_2}.(figure 2) where n_1, n_2 are respectively the order of I_{n_1} and I_{n_2}. $I_{nv}(s)$ is defined by ω_{b_i}, ω_{h_i} which represent a limited band where fractional order acts, with a total $2N$ number of cells. ω_p is an intermediate frequency coresponding to

$$n=\begin{cases}n_1 & \text{for}\quad \omega<\omega_p\\ n_2 & \text{for}\quad \omega>\omega_p\end{cases}$$

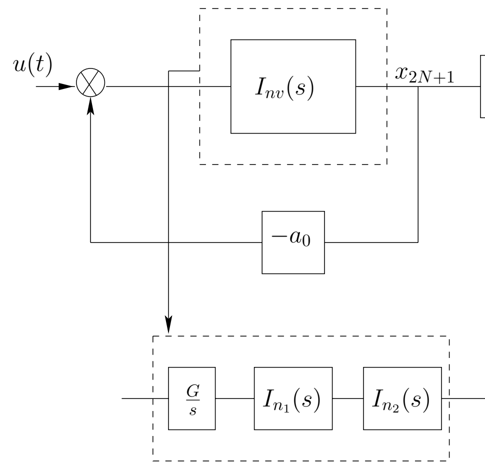

Fig. 2. simplified model

The Bode diagram of the new integrator $I_{nv}(s)$ is represented by figure 3.

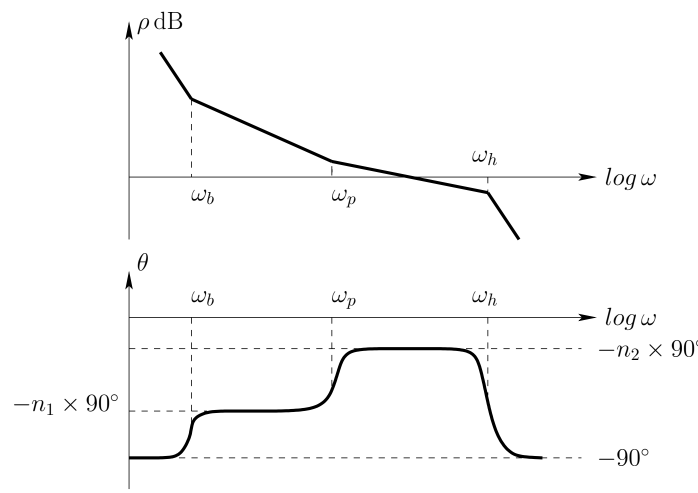

Fig. 3. Bode diagram of $I_{nv}(s)$

Remark : I_{n_1} and I_{n_2} are defined as a fractional phase-lead filter (see figure 2)

$$I_{n_k}\left(s\right)=\prod_{i=1}^{N}\frac{1+\frac{s}{\omega'_{i,n_k}}}{1+\frac{s}{\omega_{i,n_k}}} \tag{5}$$

The state-space representation of $H_{nv}(s)$ is :

$$M_I\underline{\dot{x}}_I=A_I\,\underline{x}_I+\underline{B}_I\,u \tag{6}$$

with

$$M_I = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ -\alpha_1 & 1 & 0 & \cdots & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ & \ddots & -\alpha_2 & 1 & & \vdots \\ & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & & -\alpha_2 & 1 \end{bmatrix} \quad \underline{B}_I = \begin{bmatrix} G_n \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

$$A_I = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \omega_{1,1} & -\omega_{1,1} & & & \vdots \\ 0 & \ddots & \ddots & & 0 \\ & \ddots & \omega_{1,2} & -\omega_{1,2} & \\ & & & \ddots & \ddots \\ 0 & \cdots & \cdots & \omega_{N,2} & -\omega_{N,2} \end{bmatrix} \quad \underline{x}_I = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N+1} \\ \vdots \\ x_{2N+1} \end{bmatrix}$$

with

- $\omega_{i,1}$: corresponding pulsation for the operator I_{n_1}
- $\omega_{i,2}$: corresponding pulsation for the operator I_{n_2}

C. state-space model of $H_{nv}(s)$

The model $H_{nv}(s)$ corresponds to a differential equation, with $n = f(\omega_p)$:

$$\frac{d^n y(t)}{dt^n} + a_0\, y(t) = b_0\, u(t) \tag{7}$$

Let us define $x(t)$ such as

$$X(s) = \frac{1}{s^n + a_0} U(s) \tag{8}$$

Thus, we obtain a "macro" state-space representation of this system.

$$\begin{cases} \frac{d^n x(t)}{dt^n} & = & -a_0\, x(t) + u(t) \\ y(t) & = & b_0\, x(t) \end{cases} \tag{9}$$

or equivalently using $I_{nv}(s)$

$$\begin{cases} \dot{x}_1 & = & G_n \left(-a_0\, x_{N+1} + u\right) \\ y & = & b_0\, x_{N+1} \end{cases} \tag{10}$$

The global model is :

$$\begin{cases} \underline{\dot{x}} & = & A\, \underline{x} + \underline{B} u \\ y & = & \underline{C}^T\, \underline{x} \end{cases} \tag{11}$$

with

$$\begin{cases} A & = & A_I^* - a_0\, \underline{B}_I\, \underline{C}_I^T \\ \underline{B} & = & \underline{B}_I^* \\ \underline{C}^T & = & b_0\, \underline{C}_I^T \end{cases}$$

D. Output-Error identification of the fractional model $H_{nv}(s)$

The model $H_{nv}(s)$ is in continuous time representation, thus it is preferable to use an Output-Error technique (OE) to estimate its parameters [13], [16]. The state-space model of the non integer system is :

$$\begin{cases} \underline{\dot{x}} & = & A(\underline{\theta})\, \underline{x} + \underline{B}(\underline{\theta})\, u \\ y & = & \underline{C}^T(\underline{\theta})\, \underline{x} \end{cases} \tag{12}$$

where $\underline{\theta}^T = \begin{bmatrix} a_0 & b_0 & \alpha_1 & \omega_p \end{bmatrix}$

Remark : the fractional order $n = f\{n_1, n_2\}$ is characterized by α_1, η_1, α_2, η_2, ω_b, ω_h. In practice, ω_b, ω_h, N and n_2 are imposed; then, it is sufficient to estimate α_1 and ω_p in order to estimate n.

Let us suppose that we have K data pairs $\{u_k, y_k^*\}$ where $t = k\,T_e$ (T_e : sampling period); y_k^* : noised measurement of the exact output y_k.

The state-space model is simulated using a numerical integration technique; thus one gets $\widehat{y}_k\left(u, \widehat{\underline{\theta}}\right)$ where $\widehat{\underline{\theta}}$ is an estimation of exact parameters $\underline{\theta}$. Then, one can construct the residuals :

$$\varepsilon_k = y_k^* - \widehat{y}_k\left(u, \widehat{\underline{\theta}}\right) \tag{13}$$

The optimal value of $\widehat{\underline{\theta}}$, that is to say $\underline{\theta}_{opt}$, is obtained by minimization of the quadratic criterion :

$$J = \sum_{k=1}^{K} \varepsilon_k^2 \tag{14}$$

Because $\widehat{y}_k$ is non linear in parameters, a non linear programming algorithm is used in order to estimate iteratively $\widehat{\underline{\theta}}$:

$$\underline{\theta}_{i+1} = \underline{\theta}_i - \left\{\left[J''_{\theta\theta} + \lambda I\right]^{-1} \underline{J}'_\theta\right\}_{\hat{\underline{\theta}} = \underline{\theta}_i} \tag{15}$$

with [13] :

- $\underline{J}'_\theta = -2 \sum_{k=1}^{K} \varepsilon_k \underline{\sigma}_{k,\underline{\theta}_i}$: gradient,
- $J''_{\theta\theta} \approx 2 \sum_{k=1}^{K} \underline{\sigma}_{k,\underline{\theta}_i} \underline{\sigma}^T_{k,\underline{\theta}_i}$: hessian,
- λ : monitoring parameter,
- $\underline{\sigma}_{k,\underline{\theta}_i} = \frac{\partial \hat{y}_k}{\partial \underline{\theta}_i}$: output sensitivity function.

This algorithm, also known as the Marquardt's algorithm [14] insures robust convergence, even with a bad initialization of $\widehat{\underline{\theta}}$, nevertheless in the vicinity of the global optimum.

Fundamentally, this technique is based on the calculation of gradient and hessian, themselves dependant on the numerical integration of the sensitivity functions $\underline{\sigma}_{k,\underline{\theta}_i}$ [16], which are equivalent to the regressors in the linear case [13].

III. Simulation example

Let us consider the heat transfer problem in a wall. A numerical simulation of the heat diffusion equation is performed using finite differences. Then, the model with simplified fractional integrator is identified using time responses provided by this simulation. In order to get realistic simulations, the following physical parameters, corresponding to brass, have been used :

$$\begin{cases} \rho = 8.522\ \ 10^3\,\mathrm{kg/m^3} : \text{density} \\ c = 0.385\ \ 10^3\,\mathrm{J/kg\ ^oC} : \text{specific heat} \\ \lambda = 111\,\mathrm{W/m\ ^oC} : \text{thermal conductivity} \end{cases}$$

A. Numerical simulation using finite differences

Let us consider a wall with a thickness L and a section S (figure 4). Divide the wall in I blocks of same thickness Δx. Then $L = I\,\Delta x$ and the i^{th} block where abscissa x is defined by $x = i\,\Delta x$.

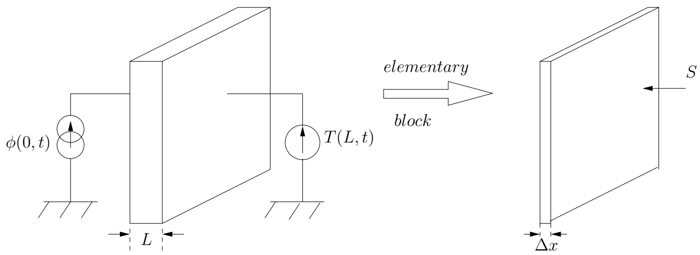

Fig. 4. The wall problem

Each elementary block is composed by :
– a thermal resistance

$$R_i = \frac{1}{\lambda}\,\frac{\Delta x}{S} \tag{16}$$

– a thermal capacity

$$C_i = \rho\, c\, S\, \Delta x \tag{17}$$

Dimensions used for the wall simulation are $L = 5$ cm and $S = 100$ cm^2. One obtains a state model with (dim $\underline{x} = I$).

$$\underline{u}^T = \left[\begin{array}{ccccc} 2\,\frac{\Phi(0,t)}{C_i} & 0 & \cdots & 0 & \frac{T(L,t)}{R_i\,C_i} \end{array}\right]$$

The obtained model is used in order to simulate the heat transfer in the wall. The time response of the system is plotted on figure 5. The input is a Pseudo-Random Binary Signal. $I = 300$ blocks are used in order to perform the simulation. Subsequently, the simulated output will be considered as the real output of the system $H(s)$.

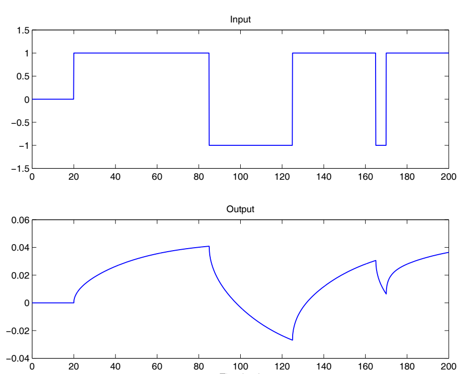

Fig. 5. Input and Output response of the $H(s)$ with $I = 300$

B. Fractional models identification

Using data provided by the previous heat transfer simulator, OE identification of the model $H_{nv}(s)$ is performed. Results are given in table I.

TABLE I
IDENTIFICATION RESULTS

Parameters	$H_{nv}(s)$
a_0	0.1286
b_0	0.0058
n_1	0.9976
ω_p	0.0997

According to figure 6, the frequency response of this model is compared with the exact frequency response of $H(s)$. The approximation obtained with the model $H_{nv}(s)$ is accurate in intermediate and low frequency. The improved model is able to explain the frequency behaviour of the simulator $H(s)$ (in its validity domain). According to figure 7, the improved model is also able to approximate very well the time response, particularly during transients. Nevertheless, we can notice that the estmated order is close to 1. Thus it is possible to simplify this frequency varying order integrator by imposing $n_1 = 1$ (and $n_2 = 0.5$) and adjusting the remaining parameters.

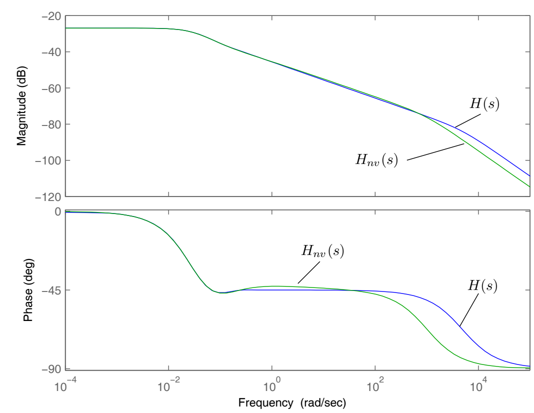

Fig. 6. Bode plots of models $H(s)$, $H_{nv}(s)$

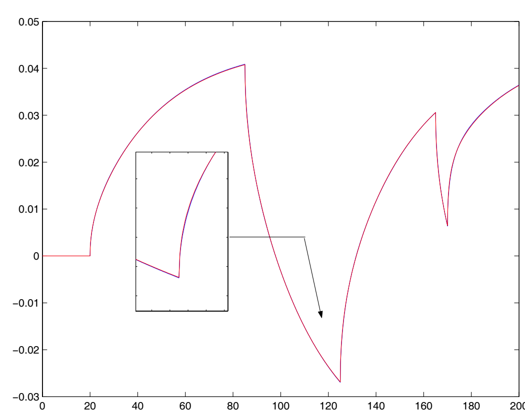

Fig. 7. Time responses of models $H(s)$, $H_{nv}(s)$

IV. Approximation of diffusive interfaces with a simplified integrator

According to the results obtained for the previous model $(H_{nv}(s))$, a simplified fractional integrator is now used. This model will enable us to get a good approximation of the diffusive interface with a restricted number of parameters $(\{a_0, b_0, \omega_p\})$.

A. Comparative simulated example

In the same way as for the first model, an output-error identification of the new model $H_{ns}(s)$ is performed using data provided by the previous heat transfer simulator. Results are given in table II.

TABLE II
Identification results

Parameters	$H_{ns}(s)$
a_0	0.1288
b_0	0.0058
ω_p	0.0982

The identification results are close to those obtained by using the model where n is variable with the frequency.

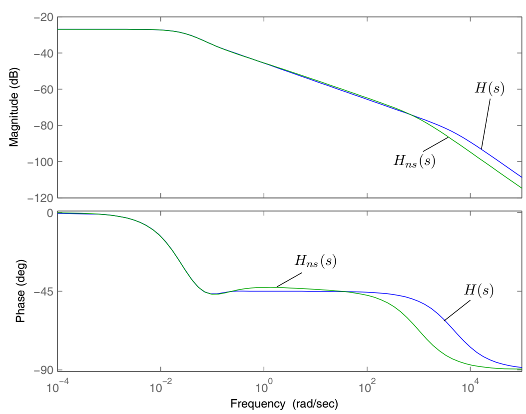

Fig. 8. Bode plots of models $H(s)$, $H_{ns}(s)$

One can notice that the new integrator enabled us to reproduce the dynamics of the simulator at low and intermediate frequencies (see figure 8) For high frequencies $(\omega > 10^3 rd/s)$ the asymptotic behaviour is $n = 1$ thanks to integer nature of the simulator. one can notice that the fractional behaviour is exhibited only for $\omega < 10^3 rd/s$ (in its field of validity). The accuracy of this new model appears perfectly in the time domain (figure 9) for the identification example and (figure 10) for the cross validation ; it is able to reproduce quick transients with excellent approximation, using a restricted number of estimated parameters.

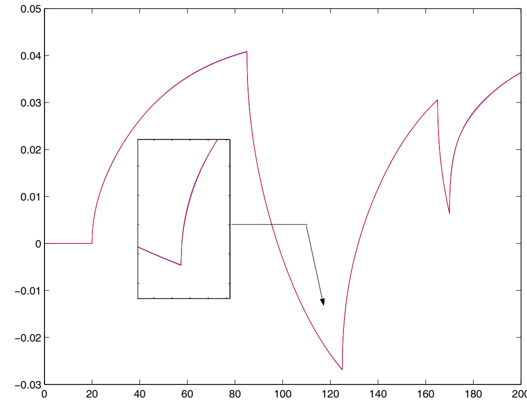
Fig. 9. Time responses of models $H(s)$, $H_{ns}(s)$-data used for identification

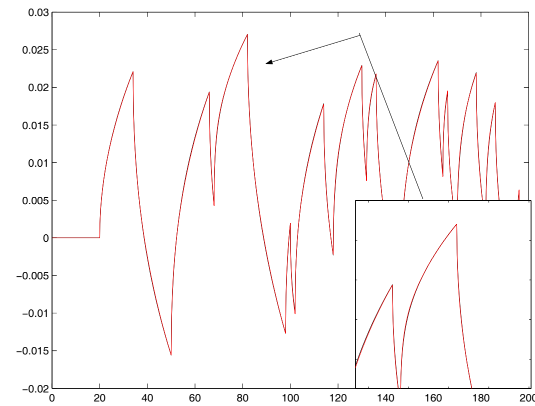
Fig. 10. Time responses of models $H(s)$, $H_{ns}(s)$-data used for cross validation

V. Conclusion

The model proposed in this paper represents a new contribution to the modelling of diffusive interfaces. Using frequency considerations, the objectif was to improve the dynamical behaviour of the model. In preceding works [3], [4], [5], [6], the modelling of diffusive systems using non integer model, with the help of a fractional integrator operator has shown its efficiency.

A theoretical approach showed that fractional modelling should be able to reproduce the essential characteristic of the diffusive phenomenon, i.e. that $n = 0.5$ then $\omega \to \infty$ while taking into account of the phenomenon geometry. The major problem is the frequency response of the diffusive interface and particularly its phase. The proposed solution in this paper consists to improve our preceding works thanks to a new model where the order n is variable with the frequency $(H_{nv}(s))$. The results revealed that this model is perfectly able to approach the system at low and medium frequencies. According to the estimated parameter $n_1 = 1$, this model $(H_{nv}(s))$ has been simplified into a new model $(H_{ns}(s))$ in imposing n_1 to 1. Simulations demonstrate that this simplified integrator is able to provide an excellent approximation of diffusive interfaces, either in time or frequency domain. Moreover, we can conclude that

this simplified fractional integrator is preferable to other previous integration operator, thanks to its occuracy and its parsimony.
The modelling of non linear diffusive interfaces is a more complex problem than the linear case. Using a neural network and this simplified fractional integrator, one can model fractional non linear sysems with a realistic number of estimated parameters [7].

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