Taking into Account of Non-Linearities in the CRONE approach: Application to Vibration Isolation

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***Abstract* – The interest of fractional differentiation in vibration isolation has been shown in many previous papers, in particular in the case of suspension achievements in hydropneumatic technology. This paper aims to show that the hydropneumatic components non-linearities do not deteriorate the remarkable performances of a particular CRONE suspension (suspension whose force-deflection transfer function is a band-limited fractional differentiator) realized in hydropneumatic technology and to propose a method to analyse these non-linearities influence. The suggested method to take into account the non-linearities is based on the Volterra series representation.**

I. INTRODUCTION

Vibration isolation consists in limiting the vibration transmission between a source of vibration and a device to isolate by adding a vibration isolator, also called suspension.

In vibration isolation with a common suspension, and in all mechanical structures in general, it can be easily shown that a sprung mass augmentation leads to the reduction of the system stability degree, or in other words to the damping reduction. The CRONE approach [1], based on the used of the fractional differentiation for the synthesis of a feedback loop controller, allows to design suspensions so that the system stability degree is maintained in spite of mass variations [2]. This suspension, called CRONE suspension, can be designed thanks to the CRONE control methodology because the suspension device plays the same role as the controller of a control loop [3].

The suspension device can be realized from various means: in mechanical technology, springs and dampers are respectively used as capacitive and dissipative elements; these two functions can also be realized thanks to visco-elastic material or thanks to hydraulic accumulators and resistors in hydropneumatic technology. This paper only deals with a particular CRONE suspension realized in hydropneumatic technology [4]. Such a suspension allows to obtain the stability degree robustness thanks to the CRONE control methodology but also the rapidity robustness. Indeed, the hydropneumatic accumulators equivalent stiffness's depend on the sprung mass value and when the mass varies, the controller and the plant variations compensate each other so that the cross-over frequency does not depends on the sprung mass value. These remarkable properties have been illustrated in [4]. However, in [4], although the hydraulic accumulators are non-linear, the design method and the simulations have been made with linearized behaviour.

The aim of this paper is to study and to explain, from an illustration example, the behaviour of a CRONE suspension when the components non-linearities are taken into account.

II. FRACTIONAL DIFFERENTIATION IN VIBRATION ISOLATION

A. *Vibration isolation principle*

Figure 1 presents a one degree of freedom model which is often used for basic studies of vibration isolation. From this model, some previous works [3] have shown that a suspension which develops a force $u(t)$ that is linked to its deflection $z_{10}(t)$ by the relation:

$$U(s) = -D(s)\,Z_{10}(s)\,, \tag{1}$$

naturally makes a feedback control around the static equilibrium position.

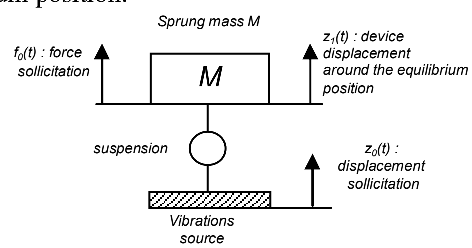

Fig. 1. One D.O.F. model

The block diagram (figure 2) which comes from the modeling clearly shows that the suspension plays the same role as the controller of a control loop and that the displacement and force solicitations $(z_0(t),\ f_0(t))$ can be considered as input and output perturbations which act on the plant. The plant is a double integrator whose transitional pulsation depends on the sprung mass.

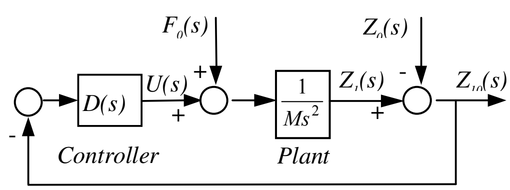

Fig 2. 1 DOF model block diagram

The open loop transfer function $\beta(s)$ is:

$$\beta(s) = D(s)\,G(s)\,,\ \text{with}\ G(s) = \frac{1}{M\,s^2}\,. \tag{2}$$

Thus, suspension design can be made by using the classic controller synthesis methods. In particular, when parameters uncertainties (especially mass variation) are considered, the

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problem becomes a robust control synthesis problem.

B. *CRONE command and CRONE suspension*

When only sprung mass value variations are considered, the use of the CRONE control methodology to design a suspension leads to a force-deflection transfer function *C(s)* which is a band limited fractional differentiator, namely:

$$C(s) = D_0 \left(\frac{1+\dfrac{s}{\omega_b}}{1+\dfrac{s}{\omega_h}} \right)^m , \qquad (3)$$

where D_0 is the static gain, ω_b and ω_h are the low and high transitional pulsations (ω_h is much greater than ω_b) and where m is the fractional order (that means that m may be not non-integer).

D_0, m, ω_b and ω_h are in the CRONE method the high level synthesis parameters. These four parameters are determined from the specification sheets. The synthesis method to obtain these four parameters can be found in [3]. The interested reader can find more detail about the CRONE suspension in [1][2][3] and [4]

B. *CRONE suspension achievement in hydropneumatic technology*

The hydraulic components used in hydropneumatic technology are hydraulic accumulators (capacitive components, *C*, which contain oil and gas separated by an impermeable diaphragm, they play the same role as mechanical spring) and hydraulic resistors (dissipative components, *R*).

The hydraulic accumulators have non-linear behaviour. Indeed, the gas pressure $p_i(t)$ in an accumulator is linked to the oil volume $v_{li}(t)$ which has entered in the accumulator by the non-linear relation

$$p_i(t) = \frac{P_s}{\left(1 - \dfrac{P_s}{P_{0i}\,V_{0i}}\, v_{li}(t)\right)^{\gamma}} , \qquad (4)$$

where P_s is the static pressure in the accumulator, i.e. the gas pressure when the system is at rest and motionless, P_{0i} and V_{0i} are the calibration pressure and the volume of the accumulator, γ is a thermodynamic coefficient (γ=1 for isotherm transformation, γ=1.4 for adiabatic transformation). This relation comes from the thermodynamic laws which describe the evolution of gas.

In the general case, the hydraulic resistors are also non-linear. However it is possible to design the suspension so that the resistors have linear behaviour, and so the pressure drop $\Delta p_i(t)$ between the i^{th} resistor input and output is proportional to the oil flow $q_i(t)$ in the resistor:

$$\Delta p_i(t) = R_i\, q_i(t) , \qquad (5)$$

where R_i is the resistance of the i^{th} hydraulic resistor.

Figure 4 presents an achievement of a CRONE suspension in hydropneumatic technology for a hydraulic test bench.

The test bench allows to study the free evolution of a mass (*M*) after a release test. The mass is mechanically linked to a hydraulic simple effect jack. The minimal mass of 75 kg can be increased by additional masses. So, *M* can vary between 75 kg and 150 kg. The suspension jack is connected to a two parts hydraulic circuit. The first part is composed of a pump equipped with a make and brake circuit and a proportional valve. Its aim is to maintain the mass *M* at a fixed height, independently of the mass value thanks to a control feedback. The second is composed of a change over valves which allows to select either a parallel arrangement of two cells including one RC cell (*N*=1, *N* is the number of RC cells), or a parallel arrangement of six cells including five RC cells (*N*=5). This arrangement constitutes a CRONE suspension.

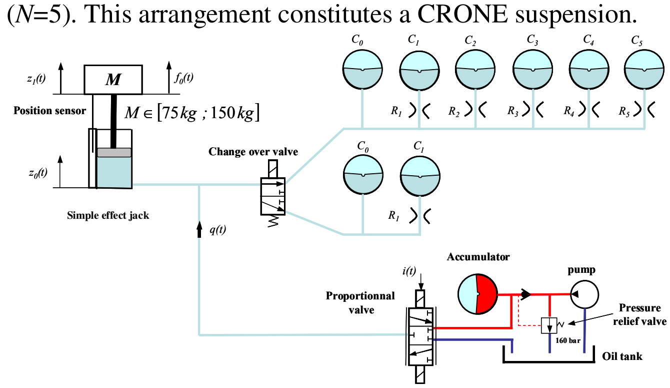

Fig. 3. Hydraulic diagram of the CRONE test bench

The associated control diagram is presented figure 4. The external loop which regulates the static equilibrium position at a value equal to half stroke of the jack has the same rapidity as the self-leveller device of a hydropneumatic suspension. This rapidity is characterized by an open-loop gain cross-over frequency of 0.1 rad/s. The internal loop has a rapidity characterised by an open-loop gain cross-over frequency of 6 rad/s, the same rapidity as the vertical mode of a usual vehicle. So, both loops are dynamically uncoupled. That is why only the internal loop is considered in the following parts of this article.

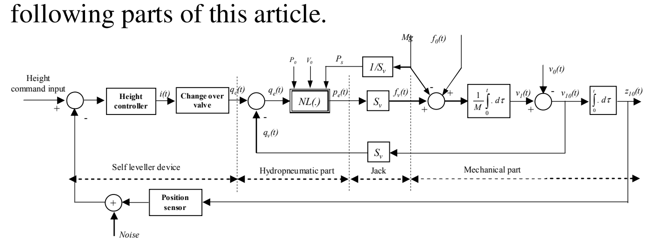

Fig. 4. Control diagram of the CRONE test bench

The functional scheme associated to the non-linear CRONE suspension is given figure 5. This scheme constitutes the validation model of the CRONE suspension.

based on Volterra series representation to analyse and explain these phenomena.

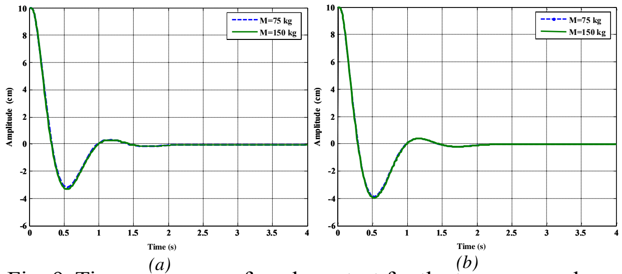

Fig. 9. Time responses of a release test for the two mass values, simulated with the linear synthesis model (a) and the non-linear validation model (b) (with a CRONE suspension)

III. TAKING INTO ACCOUNT OF COMPONENTS NON-LINEARITIES

A. Volterra series definition

Volterra series were first studied by Volterra during the 30[ies] ([5]). They were then used by Wienner ([6]) and Barret to study and analyse nonlinear systems and Fliess *et al* ([7]) for optimal control of non-linear systems. Worden, *et al.* ([8]) present Volterra series as a generalization of the well-known input-output relation for linear systems:

$$y(t)=\int_{-\infty}^{+\infty} h(t-\tau)\,u(\tau)\,d\tau\,, \tag{10}$$

where *u(t)* is the system input and *y(t)*, the output. The system is specified uniquely by its impulse response function *h(t)*.

The extended form of equation (10) was obtained by Volterra (1959) and takes the form of an infinite series:

$$y(t)=\sum_{k=1}^{\infty} y_k(t)=y_1(t)+y_2(t)+y_3(t)+\ldots\,, \tag{11}$$

where

$$y_k(t)=\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{\infty} h_k(\tau_1,\ldots,\tau_n)\prod_{i=1}^{n} u(t-\tau_i)\,d\tau_i\,, \tag{12}$$

or, in a more concise way:

$$y(t)=\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_k(\tau_1,\cdots,\tau_k)\prod_{i=1}^{k} u(t-\tau_i)\,d\tau_i\,. \tag{13}$$

The functions $h_k(\tau_{1,},\ldots,\tau_k)$ are generalizations of the linear impulse response function and are usually referred to as Volterra kernels. One can show that Volterra kernels are symmetrical ([9]).

Laplace and Fourrier transforms formulae can be generalized to the multivariable functions. The Laplace transform of the k^{th}-order Volterra kernel is given by:

$$H_k(p_1,\cdots,p_k)=\int_0^{\infty}\cdots\int_0^{\infty} h_k(\tau_1,\cdots,\tau_k)\,e^{-p_1\tau_1-\cdots-p_k\tau_k}\,d\tau_1\cdots d\tau_k\,, \tag{14}$$

with $p_i=\rho_i+j\delta_i$, and where

$$h_k(\tau_1,\cdots,\tau_k)=\frac{1}{(2\pi j)^k}\int_{\rho_1-j\infty}^{\rho_1+j\infty}\cdots\int_{\rho_k-j\infty}^{\rho_k+j\infty} H_k(p_1,\ldots,p_k)\,e^{p_1\tau_1++p_k\tau_k}\,dp_1\ldots dp_k\,. \tag{15}$$

In a more concise way, Volterra theories asserts that the response of a non-linear system is the sum of a linear part (the response of the first order kernel) and a non-linear part, which is the sum of the responses of superior order kernels,

B. Highlighting of the influence of non-linear parts in a CRONE suspension

Figure 10 presents the non-linear part response in the case of the CRONE suspension of the previous paragraph. This figure is obtained by the difference between the non-linear system response (figure 9 *(a))* and the response of the linearized system (figure 9 *(b))*.

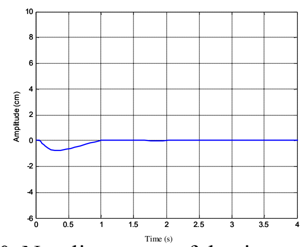

Fig. 10. Non-linear part of the time response

It is shown in the following part that this non-linear part is almost totally due to the 2[nd] order Volterra kernel.

The test bench behaviour can be described by the following equations:

- $$v_1(t)=\frac{1}{M}\int_0^t\left(-Mg+S_v\,p_e(\tau)\right)d\tau\,, \tag{16}$$

which expresses $v_1(t)$, the sprung mass vertical speed, as the result of Newton second law applied to the sprung mass ($p_e(t)$ is the oil pressure in the suspension jack),

- $$p_e(t)=p_0(t)\,, \tag{17}$$

the jack pressure is supposed to be the same as the 0[th] accumulator (the pressure drop at the jack output are neglected);

- $$p_i(t)=P_s+\frac{1}{C_i}v_{li}+\frac{1}{T_{i2}}v_{li}^{\,2}\ \ for\ i=0..N\,, \tag{18}$$

the pressure in the i[th] accumulator can be express as a polynomial function of the oil volume in this accumulator $v_{li}(t)$, this pressure is obtained by the relation (4) decomposition in Taylor series:

$$p_i(t)=P_s+\frac{\gamma P_S^{\,2}}{P_{0i}V_{0i}}v_{li}(t)+\frac{1}{2}\frac{\gamma(\gamma+1)P_S^{\,3}}{P_{0i}^{\,2}V_{0i}^{\,2}}v_{li}^2(t)+O\left(v_{li}^3(t)\right), \tag{19}$$

(it can be shown that the Taylor series can, in this case, be truncated at the second order without any significant loss of precision);

- $$\dot v_{l0}(t)=q_0(t)=q_e(t)-\sum_{i=1}^{N}\dot v_{li}(t)\,, \tag{20}$$

$\dot v_{l0}(t)$, the oil flow in the purely capacitive cell (the derivative of the oil volume which has circulated in this cell), is the difference between the oil flow which enters into the suspension $q_e(t)$ and the oil flow in all RC cells;

- $q_e(t) = S_V\left(v_0(t) - v_1(t)\right)$, (21)

the oil flow in the suspension input depends on the jack section and the deflection speed ($v_0(t)$ is the ground vertical speed);

- $q_i(t) = \frac{1}{R_i}\left(p_0(t) - p_i(t)\right)$, (22)

the oil flow in a resistance depends on the pressure drop between its input and output.

In order to establish a state-space representation of the system, these equations are re-organized:

$$\dot{v}_1 = \frac{1}{M}\left[-Mg + S_V\left(P_S + \frac{1}{C_0}v_{l0}(t) + \frac{1}{T_0}v_{l0}^2(t)\right)\right] \quad (23)$$

$$\dot{v}_{l0} = S_V(v_0 - v_1) - \sum_{i=1}^{N}\frac{1}{R_i}\left[\frac{1}{C_0}v_{lo} + \frac{1}{T_0}v_{lo}^2 - \frac{1}{C_i}v_{li} - \frac{1}{T_i}v_{li}^2\right] \quad (24)$$

$$\dot{v}_{li} = \frac{1}{R_i}\left[\frac{1}{C_0}v_{lo} + \frac{1}{T_0}v_{lo}^2 - \frac{1}{C_i}v_{li} - \frac{1}{T_i}v_{li}^2\right] \; for\ i = 1..N. \quad (25)$$

These equations can be put under the form:

$$\begin{cases} \dot{X} = A_{11}X + A_{12}X^2 + B_{10}u \\ Y = C_{11}X \end{cases} \quad (26)$$

where X is the state vector, u is the input vector and Y is the output:

$$X = \begin{pmatrix} v_1 \\ v_{l0} \\ v_{l1} \\ v_{l2} \\ v_{l3} \\ v_{l4} \\ v_{l5} \end{pmatrix}, \quad u = (v_0), \quad Y = (v_1), \quad (27)$$

and where $\otimes$ represents the tensorial product defined by:

$$A_{[m,n]} \otimes B_{[p,q]} = \begin{bmatrix} a_{1,1}B & \dots & a_{1,n}*B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \cdots & a_{m,n}*B \end{bmatrix} = C_{[m\,p,q\,n]}, \quad (28)$$

with $a_{i,j}$ is the element of the ith line and the jth column of A. This definition implies that

$$X^2 = X \otimes X = \begin{pmatrix} v_1^2 \\ v_1 v_{l0} \\ v_1 v_{l1} \\ \vdots \\ v_{l5} v_{l4} \\ v_{l5}^2 \end{pmatrix}_{49x1}, \; X^3 = X^2 \otimes X = \begin{pmatrix} v_1^3 \\ v_1^2 v_{l0} \\ v_1^2 v_{l1} \\ \vdots \\ v_{l5}^2 v_{l4} \\ v_{l5}^3 \end{pmatrix}, \quad (29)$$

In this case, the expression of A_{11}, B_{11} and C_{11} are:

$$A_{11} = \begin{pmatrix} 0 & \frac{S_V}{M\,C_0} & 0 & 0 & 0 & 0 & 0 \\ -S_V & -\frac{1}{C_0}\sum_{i=1}^{N}\frac{1}{R_1} & \frac{1}{R_1C_1} & \frac{1}{R_2C_2} & \frac{1}{R_3C_3} & \frac{1}{R_4C_4} & \frac{1}{R_5C_5} \\ 0 & \frac{1}{R_1C_0} & -\frac{1}{R_1C_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{R_2C_0} & 0 & -\frac{1}{R_2C_2} & 0 & 0 & 0 \\ 0 & \frac{1}{R_3C_0} & 0 & 0 & -\frac{1}{R_3C_3} & 0 & 0 \\ 0 & \frac{1}{R_4C_0} & 0 & 0 & 0 & -\frac{1}{R_4C_4} & 0 \\ 0 & \frac{1}{R_5C_0} & 0 & 0 & 0 & 0 & -\frac{1}{R_5C_5} \end{pmatrix}, \quad (30)$$

$$B_{11} = \begin{pmatrix} 0 \\ S_V \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad C_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (31)$$

A_{12} is a 7x49 matrix which is not given here for size issue.

[5] shows that a system represented by (26) can be put under the more compact bilinear form:

$$\begin{cases} \dot{\tilde{x}} = A\tilde{x} + D(\tilde{x} \otimes u) + Bu \\ y = C\tilde{x} \end{cases}. \quad (32)$$

In order to obtain this form, let us first consider

$$\begin{aligned} \frac{d}{dt}(X^2) &= [\dot{X} \otimes X] + [X \otimes \dot{X}] \\ &= [(A_{11}X) \otimes X + (A_{12}X^2) \otimes X + (B_{10}u) \otimes X] + \\ &\quad [X \otimes (A_{11}X) + X \otimes (A_{12}X^2) + X \otimes (B_{10}u)] \\ &= [A_{11} \otimes I_7 + I_7 \otimes A_{11}]X^2 + [A_{12} \otimes I_7 + I_7 \otimes A_{12}]X^3 \\ &\quad + [B_{10} \otimes I_7 + I_7 \otimes B_{10}](X \otimes u) \\ &= A_{22}X^2 + A_{23}X^3 + B_{21}(X \otimes u) \end{aligned} \quad ,(33)$$

where I_7 is the identity matrix of dimension 7x7.

It is important to note that the system order increases to order 3. For the calculation of dX^3/dt the result will have to be truncated. The truncation will be made at order 3 because superior order are not necessary to describe dX^2/dt.

Thus, by introducing

$$\tilde{x} = \begin{bmatrix} X \\ X^2 \\ X^3 \end{bmatrix}, \quad (34)$$

system (26) can be put under the form (32) with:

$$\begin{bmatrix}\dot{X}\\ \dot{X^2}\\ \dot{X^3}\end{bmatrix} = \begin{bmatrix}A_{11}&A_{12}&0\\0&A_{22}&A_{23}\\0&0&A_{33}\end{bmatrix}\begin{bmatrix}X\\X^2\\X^3\end{bmatrix} + \begin{bmatrix}0&0&0\\B_{21}&0&0\\B_{31}&B_{32}&B_{33}\end{bmatrix}\begin{bmatrix}X\otimes u\\X^2\otimes u\\X^3\otimes u\end{bmatrix} + \begin{bmatrix}D_{10}\\0\\\vdots\\0\end{bmatrix}u \quad (35)$$

In [10] expressions are given for the Laplace transforms of the Volterra kernels of system described by (32) as follows:

$$\begin{aligned}&P_1(s) = C(s\,I - A)^{-1}B,\\&P_2(s_1, s_2) = C[(s_1 + s_2)I - A]^{-1}D\{[(s_1 I - 1)^{-1}B]\otimes I_1\}.\end{aligned} \quad (36)$$

These expressions allow to compute the Laplace transforms of the first two Volterra kernels of the hydraulic test bench presented in the previous paragraph. They also allow to obtain the time responses of the two first Volterra kernels thanks to the realisation method described in [10, 11]. The second order Volterra kernel can be realized in the configuration shown in figure 11.

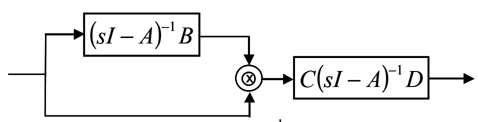

Fig. 11. Realization of the 2nd order Volterra kernel

The time response of the sum of the first and second order kernels is very close to figure 9(b), this shows that superior order contribution can be neglected in the future analysis.

Figure 12 and presents the first and second order Volterra kernels gain diagrams.

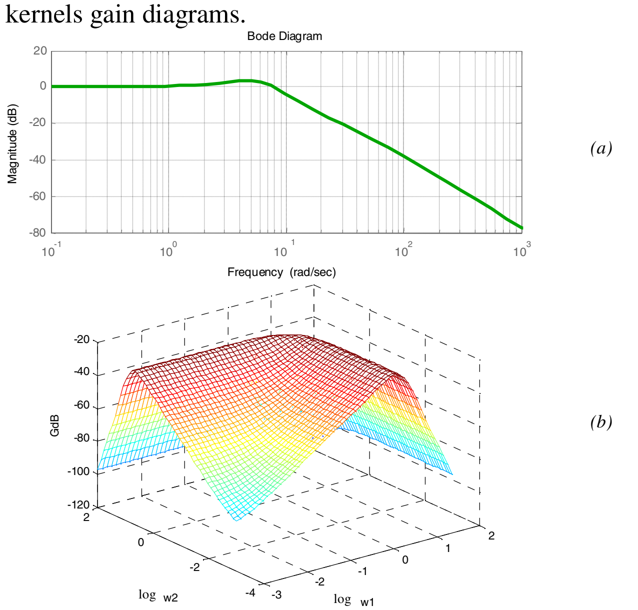

Fig. 12. Gain diagram of the test bench first (a) and second (b) order Volterra kernels

We can note that this second order kernel gain diagram tends towards -∞ when ω_1 and ω_1 tends towards 0 and +∞, that means that the non-linearities do not have any influence in low and high frequencies. The gain of the second order kernel is very low in regards to the first order kernel gain.

IV. CONCLUSION

This paper presents the beginning of our work on non-linear behaviour in suspension. The first step which is given here is the proposition of a mathematical tool to describe and analyze non-linearities.

Taking into account the non-linearities in vibration isolation is interesting for at least two reasons. First, it is a natural prolongation for the CRONE team previous works in which the non-linearities influence was considered negligible because only small variations were studied. Second, the specification sheets definition for non-linear component is an open issue for which our industrial partner is very concerned. This is our final goal. The next step toward this goal is to take into account the hydraulic resistor non-linearities.

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