Identification of fractional order models for electrical networks

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Abstract – **In this article, the authors propose a new modelling of electrical networks based on fractional order systems, i.e. systems with non-integer derivative functions. The algorithm of identification of such systems has been applied on a particular network presenting a fractal behaviour on a restricted frequency range.**

I. INTRODUCTION

Today, electrical supply networks are more and more used for distributing energy. It is worth to mention the case of automotive and aeronautic fields, where the electrification of numerous functions has led to an increasing interconnection of electrical devices. Moreover, all the electrical networks (from on-board systems to transport ones) become more and more complex, including an increasing number of power electronics converters [1].

A big issue concerning these networks is their design. Due to their complexity, the classical design process based on empirical rules is not anymore usable for reaching the objectives of efficiency, of weight and dynamic performances while respecting the safety and reliability constraints. A specific process is required mixing up modelling work, optimisation and numeric simulation.

Modelling of electrical power systems is far from being obvious, especially if dynamic performances are considered. Indeed, dynamic models may include a huge number of parameters for taking into account the numerous modes introduced by power converters, their filters and control loops. Of course, the number of parameters increases along with the size of the electrical network too.

For large power systems, many works propose to reduce the order of the model by using modal reduction of the state-space model (based on Hankel or Moore's method) or by reducing the number of characteristic parameters of the frequency response [2]. Using frequency models for power systems is attractive compared to state-space models due to several advantages. Firstly there is a lesser need for a priori determination of the model. The frequency approach naturally takes into consideration high-order dynamics that usually are not captured by a parametric approach. Secondly, frequency response methods are commonly used to analyse the behaviour of electrical systems or components and many electrical engineers know them. Thirdly, characteristics such as stability, fault conditions and bandwidth can be investigated from the frequency response plot. Finally, frequency response naturally takes into account both additive and spectral changes.

On the other hand, frequency modelling of electrical networks raises the issue of measuring in vivo the frequency response. One method of measuring it is to inject a variable frequency current into the system and measure the response of the network at each frequency [3]. This method requires special devices to inject the test currents.

In this paper, the authors will focus on an original way of frequency modelling which is based on the fractional-order theory. There are several studies which have focused on electrical machines modelling using half-order systems [4, 5, 6]. After a short definition of such systems, the authors will present an identification method dedicated to fractional-order models of electrical systems. Then, the method will be applied to a fractal network, i.e. a network which presents an arborescent topology. Indeed, this kind of network clearly exhibits a fractional-order zone in its frequency response, which seems to be a very interesting way of compact modelling.

II. THEORETICAL CONSIDERATIONS ON FRACTIONAL ORDER SYSTEMS

The α -order derivative of a function f is given by [7, 8]:

$$D^{(\alpha)} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \cdot \sum_{k=0}^{\infty} (-1)^k \cdot \binom{\alpha}{k} \cdot f(t - k \cdot h) \qquad (1)$$

where $\binom{\alpha}{k} = \frac{\Gamma(\alpha)}{\Gamma(k) \cdot \Gamma(\alpha - k)}$ (Γ is the Gamma function, defined by $\Gamma(z) = \int_0^{\infty} e^{-t} \cdot t^{z-1} dt$).

can be real or even complex.

Equation 1 is the mathematical definition of the generalised derivative. It is valid for classical derivative as for non-integer or fractional ones [7].

Reference 7 shows that for causal function, (1) can be reduced to K serial terms. The α - order derivative of f can then be sampled at each sample time $t_m = h \cdot m$, as:

$$D^{(\alpha)} f_m \approx \frac{1}{h^{\alpha}} \sum_{k=0}^{m} (-1)^k \cdot \binom{\alpha}{k} \cdot f_{m-k} \qquad (2)$$

where $f_m = f[t_m]$

This second definition is used for computing the time response of the derivative.

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A. Implicit and explicit fractional order systems

One considers a single-input/single-output fractional-order system. e and s are respectively the input and the output of the system. If the system is defined by:

$$\tau^{\alpha} D^{(\alpha)} s(t) + s(t) = e(t) \tag{3}$$

it is told to be an explicit fractional-order system as the argument of the derivative operator is s. If e and s verifies (4):

$$\tau^{\alpha} D^{(\alpha)} \cdot \left[s(t) \cdot e^{t/\tau} \right] + s(t) = e(t) \tag{4}$$

it defines an implicit system as the argument is not only s but includes the product of *s* with an exponential function (where τ is the time constant of the system). It has been show in [7] that implicit systems allow to model diffusion phenomena for finite dimension case (i.e. rotor bar of induction machine), contrary to explicit ones used for infinite dimension problem.

B. Frequency properties of fractional order systems

The Laplace transform L of the α-order derivative of a causal function f can be defined as [6, 7]:

$$L\left[D^{(\alpha)} f(t); s\right] = s^{\alpha} \cdot L\left[f(t); s\right]. \tag{5}$$

Then, explicit and implicit systems can be described by the respective transfer functions:

$$H_{explicit}(s) = \frac{1}{1 + (\tau \cdot s)^{\alpha}} \tag{6.a}$$

$$H_{implicit}(s) = \frac{1}{(1 + \tau \cdot s)^{\alpha}} \tag{6.b}$$

Fig. 1 shows the Bode diagrams of such systems for = 1s and α = 0.5.

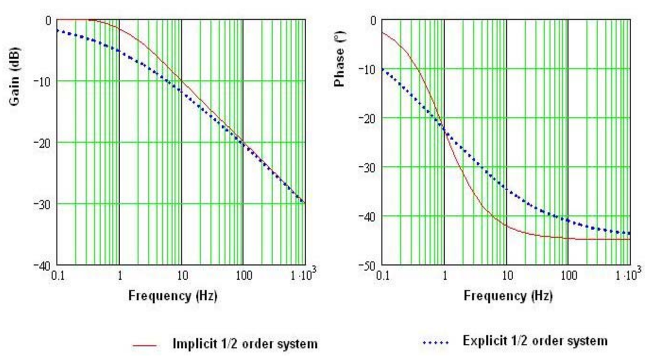


Fig. 1. Comparison between implicit and explicit half-order systems (= 1s and α = 0.5).

III. IDENTIFICATION OF THE FRACTIONAL-ORDER MODEL OF AN ELECTRICAL NETWORK

A. Presentation of the studied electrical network

By analogy with Cantor's bar [9, 10], it is possible to define a fractal electrical network. It is basically a multilayer network which every RLC cells are linked together by a scaling factor, see Fig. 2. To simplify our study, the coefficient b is taken equal to a^2.

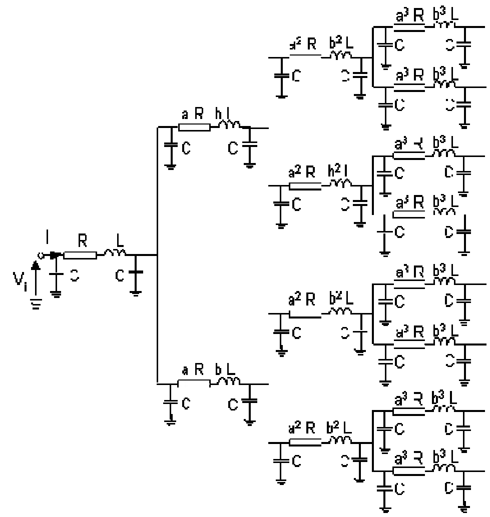


Fig. 2. Topology of the Cantor-derived electrical network (R=100 , L=1mH, C=1μF, a=3).

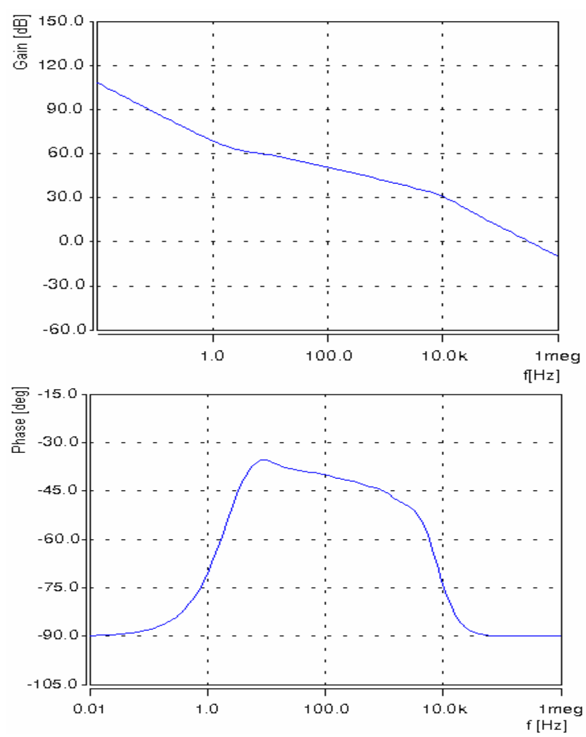


Fig. 3. Bode diagrams of the input impedance of the electrical circuit shown in Fig. 2 (R=100 , L=1mH, C=1μF, a=3).

The frequency response of the system impedance is computed by simulation, with six cells. It is shown on Fig. 3. Actually, such frequency responses can be obtained for a large range of frequencies, i.e. [0;1kHz] [11], but capacity phenomena appear up to this limit. These frequency responses can not be obtained for very high frequencies. This is not a problem because the non-integer order appears in measurable range frequency. Three distinguished zones can be clearly seen. At low-frequencies, the frequency response exhibits an inductive behaviour. At high-frequencies, the capacitances are dominant. At medium-frequencies, the impedance response is characterized by a quasi-constant phase equal to $-\eta \cdot 90\,\text{deg}$ and a gain slope of $-\eta \cdot 20$dB/decade. This is typical of fractional-order behaviour. The value of the order is then equal to η [9, 10].

B. Identification method

The objective of the method is to establish the mathematical model able to reproduce the physical behaviour of the system with a good accuracy from a series of test data.

1) Output error identification algorithm: The method used by the authors is classical. It is based on the Levenberg-Marquardt optimization algorithm associated to a quadratic logarithmic criterion based on the output error method. The whole method can be represented as following [12, 13]:

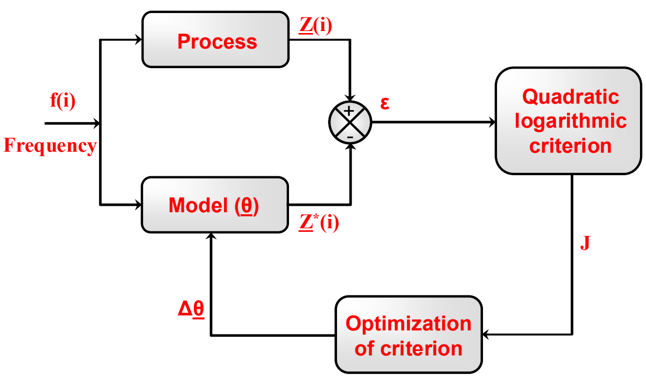


Fig. 4. Diagram of the identification method.

2) Models to be identified: The authors have chosen three kinds of models to be identified:

- a fractional-order explicit model:

$$Z(j\omega) = \frac{K_0}{j\omega} \cdot \frac{\sum_{k=0}^{2} a_k \cdot (j\omega)^{m_{ak}}}{1 + \sum_{k=1}^{2} b_k \cdot (j\omega)^{n_{bk}}}, \qquad (7)$$

where $m_{a_k} \in \Re$ and $n_{b_k} \in \Re$.

- a fractional-order implicit model:

$$Z(j\omega) = \frac{K_0}{j\omega} \cdot \prod_{k}^{4} \left(1 + \frac{(j\omega)}{\omega_k}\right)^{n_k}, \qquad (8)$$

where $n_k \in \Re$.

- an integer-order model:

$$Z(j\omega) = \frac{\sum_{k=0}^{7} a_k \cdot (j\omega)^{m_{ak}}}{1 + \sum_{k=1}^{9} b_k \cdot (j\omega)^{n_{bk}}}, \qquad (9)$$

where $m_{a_k} \in \aleph$ and $m_{a_k} \in \aleph$.

The vectors of parameters which should then be identified are respectively:

$$\underline{\theta}_{\exp} = [K_0, n_{ak}, a_k, m_{bk}, b_k];$$
$$\underline{\theta}_{imp} = [K_0, n_k, \omega_k];; \qquad (10)$$
$$\underline{\theta}_{integer} = [n_{ak}, a_k, m_{bk}, b_k].$$

The frequency range concerned by the identification goes from 10^{-2} to 10^{5} Hertz.

3) Levenberg-Marquardt optimization method: To compute the identified models, the authors have defined a criterion J depending implicitly on the parameters of the model. To minimize this criterion, the Levenberg-Marquardt algorithm is used coupling non linear programming and sensitivity functions [12]. The principle of the method can be summed up by the following recurrence equation:

$$\underline{\theta}_{i+1} = \underline{\theta}_i - \left[J''_{\theta\theta'} + \gamma_i I\right]^{-1} J'_{\theta'} \qquad (11)$$

The strictly positive parameter γ_i (Marquardt parameter) allows controlling the direction of research of the minimum of J by using mathematical relaxation. Indeed, γ_i is divided by 2 for each successful iteration. This method combines gradient (for large values of γ_i) and Gauss-Newton (for small values of γ_i) methods. $J'(\underline{\theta})$ is the gradient vector defined as the vector of partial derivative of the criterion J_{θ_i} and $J''(\underline{\theta})$ is the Hessian matrix, the Gauss approximation of $\partial J'(\underline{\theta}) / \partial \theta_i$.

This optimization method is associated to a quadratic logarithmic criterion which is easy to use for frequency response. In Fig. 4, the outputs z and z^* correspond to the Neperian logarithm of the complex impedance of the simulated process and of the mathematical model respectively. The transfer function of the complex impedance can be written as following:

$$\underline{Z}(j\omega) = Z(\omega) \cdot \exp(j \cdot \varphi(w)) \qquad (12.a)$$
$$\underline{Z}^*(j\omega, \underline{\theta}) = Z^*(\omega, \underline{\theta}) \cdot \exp(j \cdot \varphi^*(w, \underline{\theta})) \qquad (12.b)$$

The criterion $J(\underline{\theta})$ is finally given by:

$$
\begin{aligned}
J(\underline{\theta}) &= \frac{1}{2}\sum_{k=1}^{N}\left|\varepsilon_k\right|^2 \\
&= \frac{1}{2}\sum_{k=1}^{N}\left|\ln(\underline{Z}^*(j\omega_k,\underline{\theta})) - \ln(\underline{Z}(j\omega_k))\right|^2 \\
&= \frac{1}{2}\sum_{k=1}^{N}(Z^*(\omega_k,\underline{\theta}) - Z(\omega_k))^2 + \\
&+ \frac{1}{2}\sum_{k=1}^{N}(\varphi^*(\omega_k,\underline{\theta}) - \varphi(\omega_k))^2 \\
&= J_1(\underline{\theta}) + J_2(\underline{\theta})
\end{aligned}
\tag{13}
$$

The gradient and the Hessian could thus be defined similar to:

$$J'(\underline{\theta}) = \frac{\partial J_1(\underline{\theta})}{\partial \underline{\theta}} + \frac{\partial J_2(\underline{\theta})}{\partial \underline{\theta}} \tag{14}$$

$$J''(\underline{\theta}) = J_1''(\underline{\theta}) + J_2''(\underline{\theta}) \tag{15}$$

4) Identification results: The identification method is applied to the input impedance of the electrical network shown in Fig. 2. The identified models are:

- fractional-order explicit model:

$$Z_{implicit} = \left(\frac{10000}{j\cdot\omega}\right)\cdot\frac{(2.2 + 0.8\cdot(j\cdot\omega)^{0.52} - 1.4\cdot(j\cdot\omega)^{0.25})}{(1 - 0.03\cdot(j\cdot\omega)^{0.33} + 0.0065\cdot(j\cdot\omega)^{0.51})} \tag{16}$$

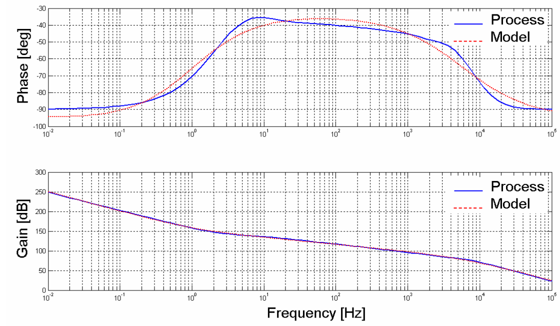
- fractional-order implicit model:

$$
\begin{aligned}
Z_{implicit} &= \left(\frac{15686}{j\cdot\omega}\right)\cdot\left(1+\frac{j\cdot\omega}{29}\right)^{-0.59}\cdot\left(1+\frac{j\cdot\omega}{29}\right)^{-0.59}\cdot \\
&\quad \cdot\left(1+\frac{j\cdot\omega}{41253}\right)^{-0.6}\cdot\left(1+\frac{j\cdot\omega}{18}\right)^{1.75} \\
&= \left(\frac{15686}{j\cdot\omega}\right)\cdot\left(1+\frac{j\cdot\omega}{29}\right)^{-1.18}\cdot \\
&\quad \cdot\left(1+\frac{j\cdot\omega}{41253}\right)^{-0.6}\cdot\left(1+\frac{j\cdot\omega}{18}\right)^{1.75}
\end{aligned}
\tag{17}
$$

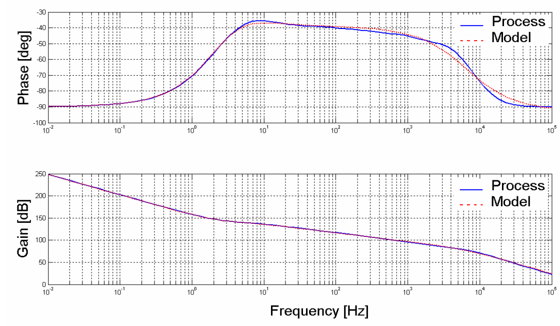
- integer-order model:

$$
Z_{\text{int}} = \frac{\begin{aligned} &5.646\cdot 10^{11}(j\omega)^7 + 5.463\cdot 10^{13}(j\omega)^6 \\ &+ 2.277\cdot 10^{14}(j\omega)^5 - 3.813\cdot 10^{14}(j\omega)^4 \\ &- 8.784\cdot 10^{14}(j\omega)^3 + 3.229\cdot 10^{15}(j\omega)^2 \\ &+ 4.287\cdot 10^{14}(j\omega) - 1.584\cdot 10^{16} \end{aligned}}{\begin{aligned} &(j\omega)^9 + 3.606\cdot 10^6(j\omega)^8 \\ &+ 2.392\cdot 10^9(j\omega)^7 + 6.151\cdot 10^{10}(j\omega)^6 \\ &+ 3.892\cdot 10^{10}(j\omega)^5 - 4.519\cdot 10^{11}(j\omega)^4 \\ &+ 3.378\cdot 10^{11}(j\omega)^3 + 2.442\cdot 10^{12}(j\omega)^2 \\ &- 6.269\cdot 10^{12}(j\omega) + 2.018\cdot 10^7 \end{aligned}}
\tag{18}
$$

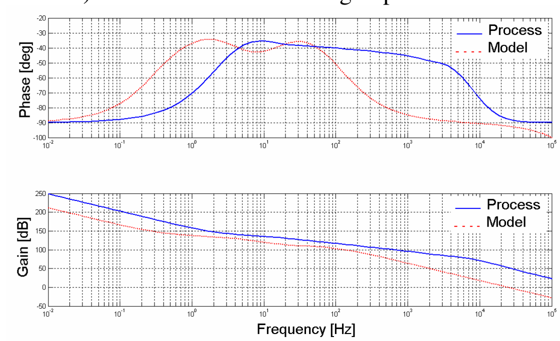
And the results are given on fig. 5. a), b) and c).



a) identification results using explicit model.



b) identification results using implicit model.



c) identification results using integer-order model.

Fig. 5. Bode diagrams of the input impedance of identified models.

The identification results show a relatively good accuracy for the fractional-order models and a poor accuracy for the integer one. Of course, an integer model could give such good results, but a very high number of parameters should be used for taking into account the fractality of the network. In our example, this number would be of 24 for the numerator (the degree of numerator) and 26 for the denominator.

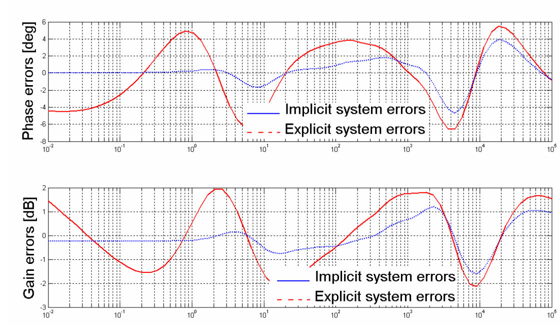


Fig. 6. Identification errors of the fractional-order models.

In Fig. 6, it clearly appears that the implicit fractional-order model is more accurate than the explicit one, especially at low frequencies. The authors assume that this difference comes from the finite geometrical size of the fractal network. This difference has been also observed for diffusion modelling in deep bar of electrical machines [6].

It should be noted that the identified parameters of the implicit model could not be linked to the fractal dimensions of the network. This is only due to the chosen structure of the model which does not represent the typical Constant Phase Angle behaviour at medium frequencies [9].

IV. CONCLUSIONS

This paper presents a frequency domain method to model and identify the input impedance of a distribution electrical network derived from the fractal of Cantor.

It was employed in order to identify the input impedance of an electrical network, using two fractional order models and an integer order model. The best result was obtained with the implicit fractional order model.

The identification results show us that the electrical networks can be represented by reduced-size models. Of course, in the near future, the authors will confirm the results by investigating real distribution networks.

Possible applications of this modelling are stability or fault location studies which require accurate models over a large frequency range. In both cases, the authors think that fractional-order models would reduce the computation cost of the analysis and would make possible to link geometrical sizes of the network and model parameters. In case of stability investigation, fractional approaches could give some element of geometrical solutions for improving the stability margin. For fault location, the authors think that the fractional model would include in its own parameters an idea of the location of the fault.

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