Non-normal diffusion and boundary conditions

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Mass spreading in heterogeneous porous media: Space fractional variants of Fourier's law

. **extreme events, heavy tails**
. **non-symmetric distributions**

stable Lévy laws: very robust/ Σ of independent random variables

the challenge: skewness + boundary data

: mass flux for Lévy flights

organization

1. Lévy flights
2. Macroscopic level: infinite domain
3. Mass flux for Lévy flights
4. Reflexive wall
5. Soft wall

small scale models

possibly non-symmetric Lévy flights

1. Lévy flights

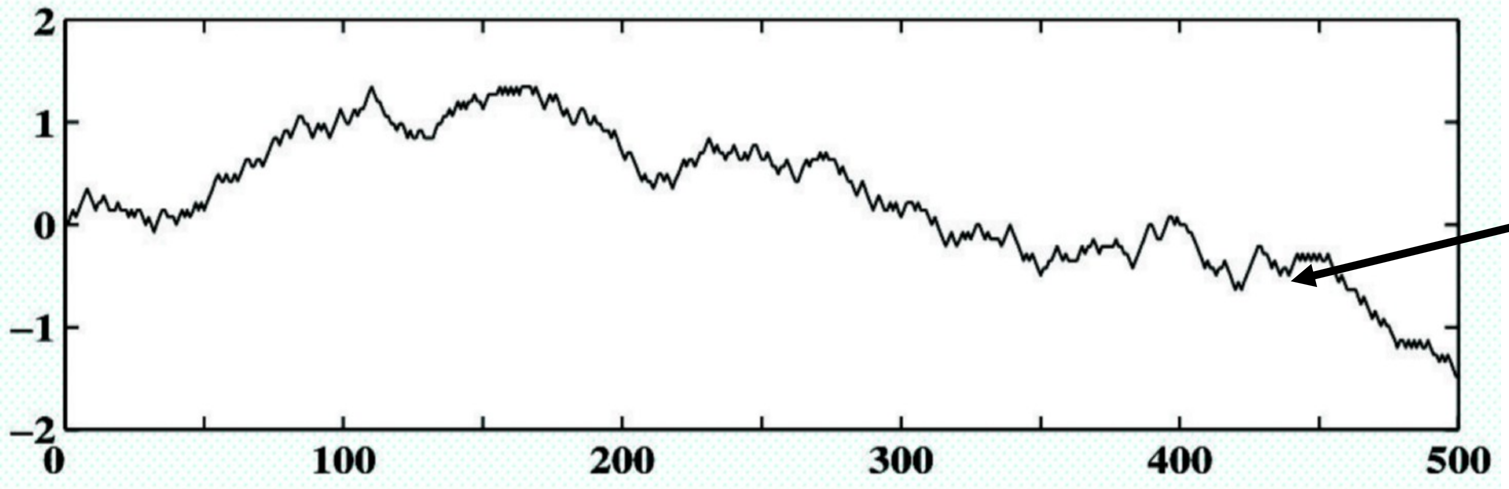
1.a. Brownian motion / Fourier's law
1. b. Alpha stable Lévy laws
1.c. Random walks, generalizing Brownian motion

1.a. Heat equation and Brownian motion

$$\partial_t C(x,t) = K \nabla_x^2 C(x,t)$$

Microscopic model: Brownian motion

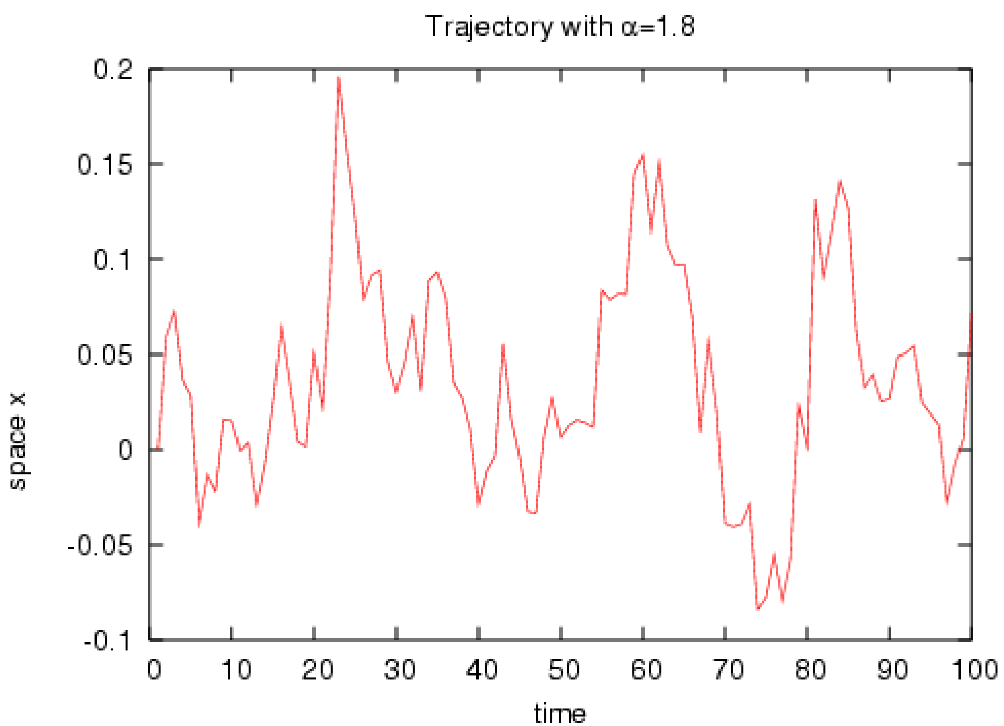
At n **τ**, each particle jumps *l* **x , normally distributed R.V.**

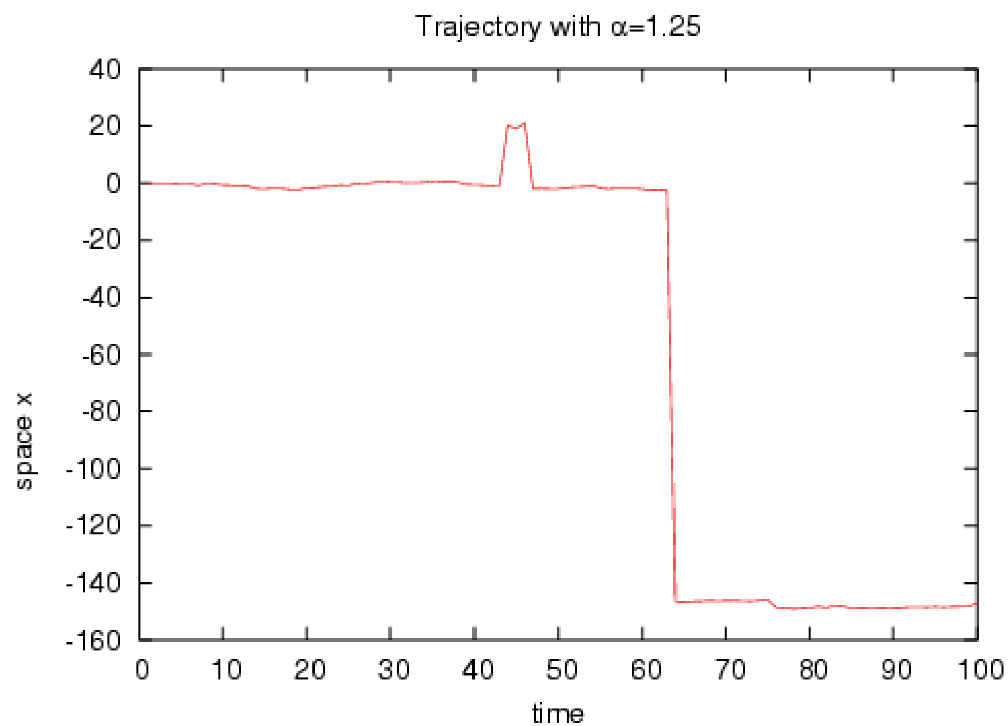


Trajectory of a Brownian motion

For transport is a disordered media? Lévy Flights give more importance to extreme events

Walkers perform successive independent jumps, whose length is distributed according to an α stable Lévy law, $\alpha < 2$





1.b. Alpha stable Lévy laws

invariant under addition of R.V., up to dilatations and translations

Definitions: X, R.V., distributed according to F: $P\{X \le x\} = F(x)$

F(x) and F(ax+b) are "of the same type"

F is **stable** when, each time 2 R.V with laws of the same type as F are added, we obtain a law of the same type

$$F(a_1 x + b_1) * F(a_2 x + b_2) = F(ax + b)$$

Fundamental property of stable laws: attractors

Definition: let F be the law of the X_n *independent R. V.*

the law G is an attractor for F if there exists sequences A_n B_n

such that the law of $\frac{(X_1 + \ldots + X_n)}{B_n} - A_n$ tends to G when $n \to \infty$

each stable law has a non empty attraction domain

Asymptotics

if F is stable, then we have 2 possibilities for $x \to \infty$

$F(-x)$ proportional to $x^{-\alpha}$ for $x>0$

,**density** proportional to $x^{-\alpha-1}$

for $0<\alpha<2$

For $\alpha=2$, density e^{-x^2/σ^2}

α: stability index of F

Σ *independent identically distributed R.V. behaving as* $x^{-\alpha-1}$ *when* $x \to \infty$

suitable power of n

stable law of index $\boldsymbol{\alpha}$

for $0<\alpha<2$

normal law

for $\alpha \geq 2$

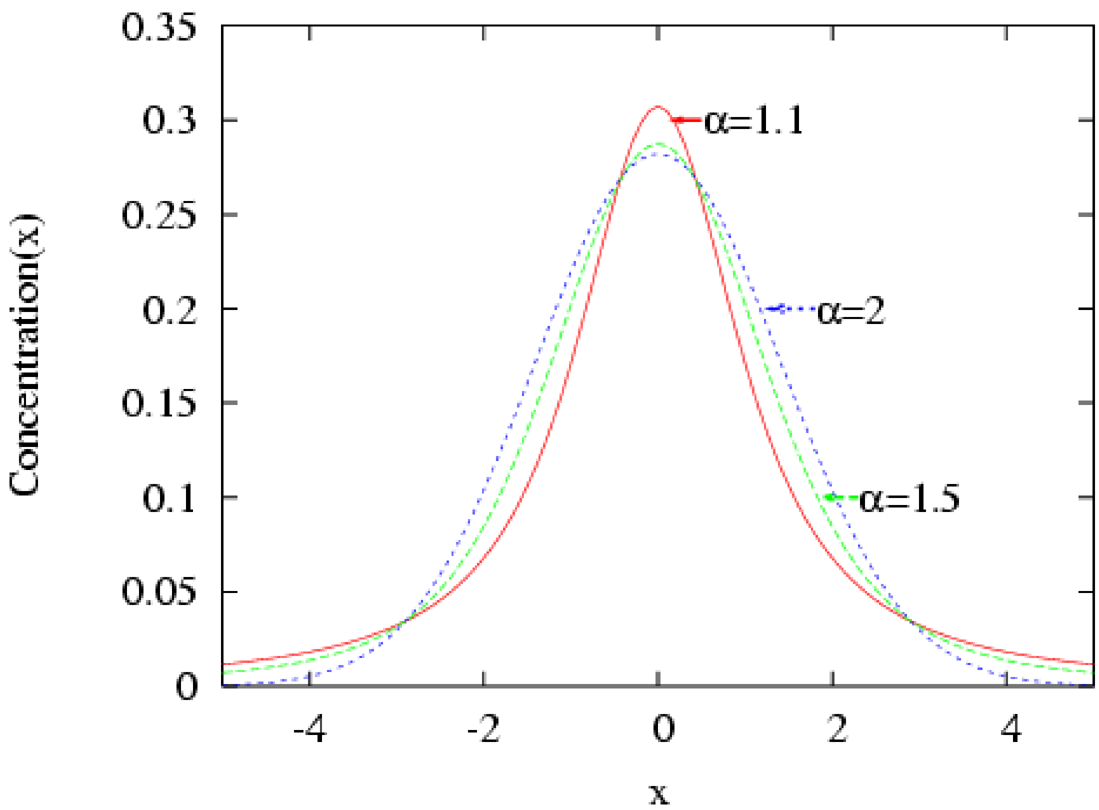
Densities of stable laws, indexed by $\alpha \in [0,2]$

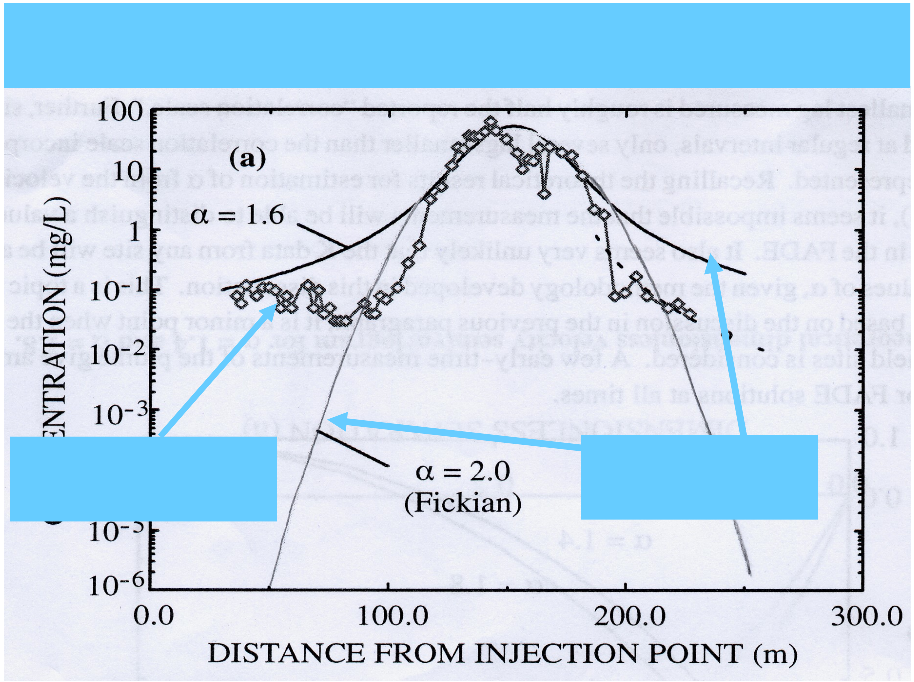
LOGARITHM OF THE Fourier transform of the density:

$-|k|^{\alpha} e^{isg(k)\theta\pi/2}$ particular case $\theta = 0$: symmetric law $-|k|^{\alpha}$

in physical variables: except for ***α****=2, the density behaves asymptotically as*

$$|x|^{(-\alpha-1)}$$





Symmetric and skewed Lévy laws

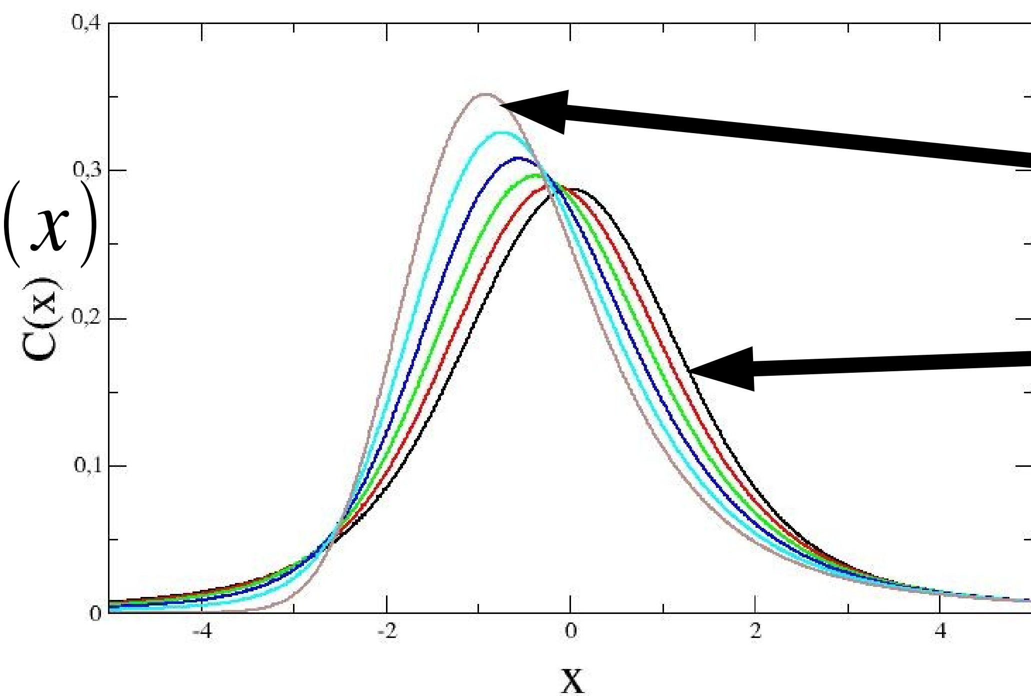
Fourier transform of the density: $e^{-|k|^{\alpha}[e^{i\,sign(k)\theta\pi/2}]}$

θ : skewness parameter

$|\theta| < 2-\alpha$ → only symmetric stable law for $\alpha = 2$

Densities of skewed stable laws for $\alpha = 1.5$

$p_{\alpha}^{\theta}(x)$



$\theta = \alpha - 2$

$\theta = 0$

1.c. Random walks, more general than Brownian motion

A collection of walkers.
The trajectory of 1 walker: a succession of instantaneous jumps, separated by
randomly distributed waiting times. Jump lengths are random also.

Jump length: lX, Waiting times between 2 successive jumps : $\tau \Phi$

α stable Lévy law p_{α}^{θ}

Let l ***and*** τ ***tend to 0 while satisfying*** $l^{\alpha}/\tau = K$

For $\alpha = 2$ (Brownian motion) the concentration of walkers
Heat equation

For $\alpha < 2$(Lévy flights) the concentration of walkers satisfies an other equation

with fractional derivatives instead of Laplacean

2. Macroscopic models for non-normal spreading

Evolution of the concentration of a cloud of walkers, performing Lévy flights?

2.a Fractional Integrals and derivatives

2.b. Fractional PDE generalizing Heat equation

2.a Fractional Integrals and derivatives

Riemann-Liouville-Weyl fractional integrals for $\alpha > 0$

$$\boxed{I^{\alpha}_{-\infty,x} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-y)^{\alpha-1} f(y)\, dy}$$

$$\boxed{I^{\alpha}_{x,+\infty} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} (y-x)^{\alpha-1} f(y)\, dy}$$

2 Derivatives on both sides of x: n integer, n-1<α<n

$$\left(D^{\alpha}_{-\infty,x} I^{\alpha}_{-\infty,x} f\right)(x) = f(x)$$

$$D^{\alpha}_{x,+\infty} I^{\alpha}_{x,+\infty} f = f$$

Left inverse of

$$D^{\alpha}_{-\infty,x} f(x) = \frac{d^n}{dx^n} I^{n-\alpha}_{-\infty,x} f(x)$$

$$\frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^{x} (x-y)^{n-\alpha-1} f(y)\, dy$$

Left inverse of

$$D^{\alpha}_{x,+\infty} f(x) = (-1)^n \frac{d^n}{dx^n} I^{n-\alpha}_{x,+\infty} f(x)$$

$$\frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{x}^{+\infty} (y-x)^{n-\alpha-1} f(y)\, dy$$

2.b. Evolution of the concentration

A collection of walkers.

The trajectory of 1 walker: a succession of instantaneous jumps, separated by randomly distributed waiting times. Jump lengths are random also.

Jump length: $l X$ Waiting times between 2 successive jumps : $\tau \Phi$

α stable Lévy law, p_{α}^{θ} possibly skewed

If l and τ tend to 0, with $l^{\alpha} / \tau = K$

the concentration of walkers satisfies

$$\boxed{\partial_t C(x,t) = K \nabla_{\theta}^{\alpha} C(x,t)}$$

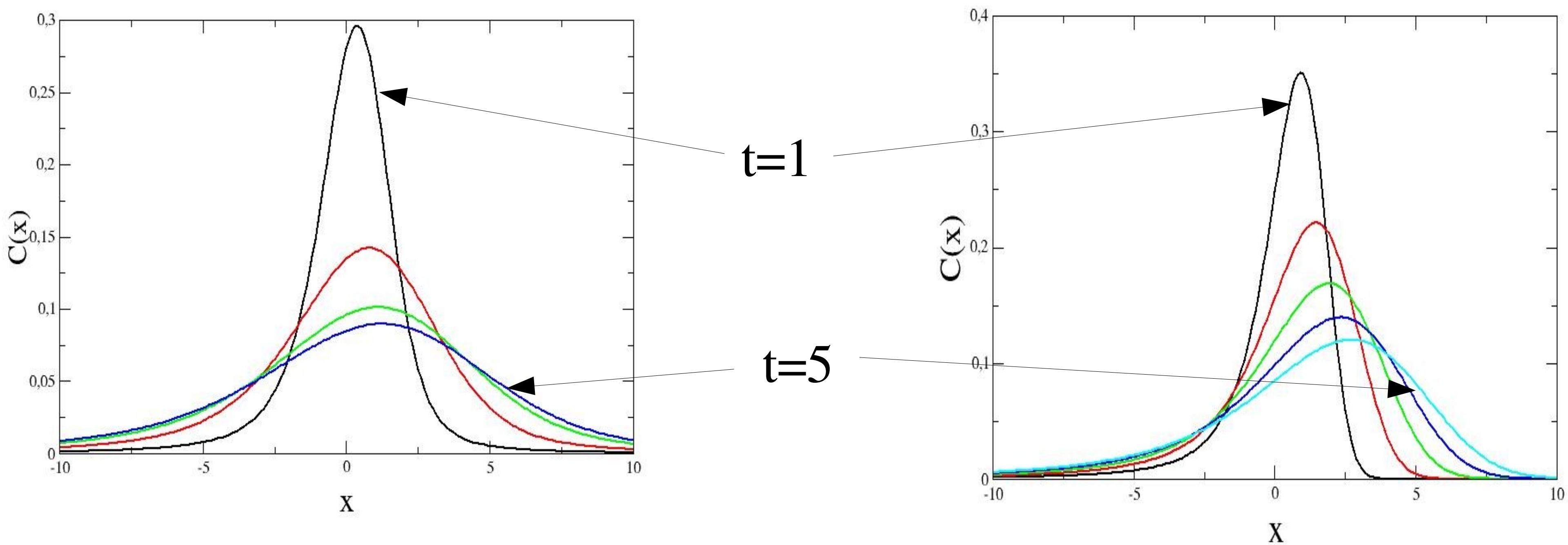
Riesz-Feller derivative, in its symmetrical ($\theta = 0$) or skewed version

$$\boxed{\nabla_{\theta}^{\alpha} = \frac{-1}{\sin(\alpha \pi)} \left(\sin(\pi/2(\alpha - \theta)) D_{-\infty, x}^{\alpha} + \sin(\pi/2(\alpha + \theta)) D_{x, \infty}^{\alpha} \right)}$$

Fourier symbol: $-|k|^{\alpha} [e^{i\, sign(k) \theta \pi/2}]$

Fundamental Solutions: $C(x,t) = p_{\alpha}^{\theta}\left(x/(Kt)^{1/\alpha}\right)/(Kt)^{1/\alpha}$

density of α *stable Lévy law with skewness parameter* θ



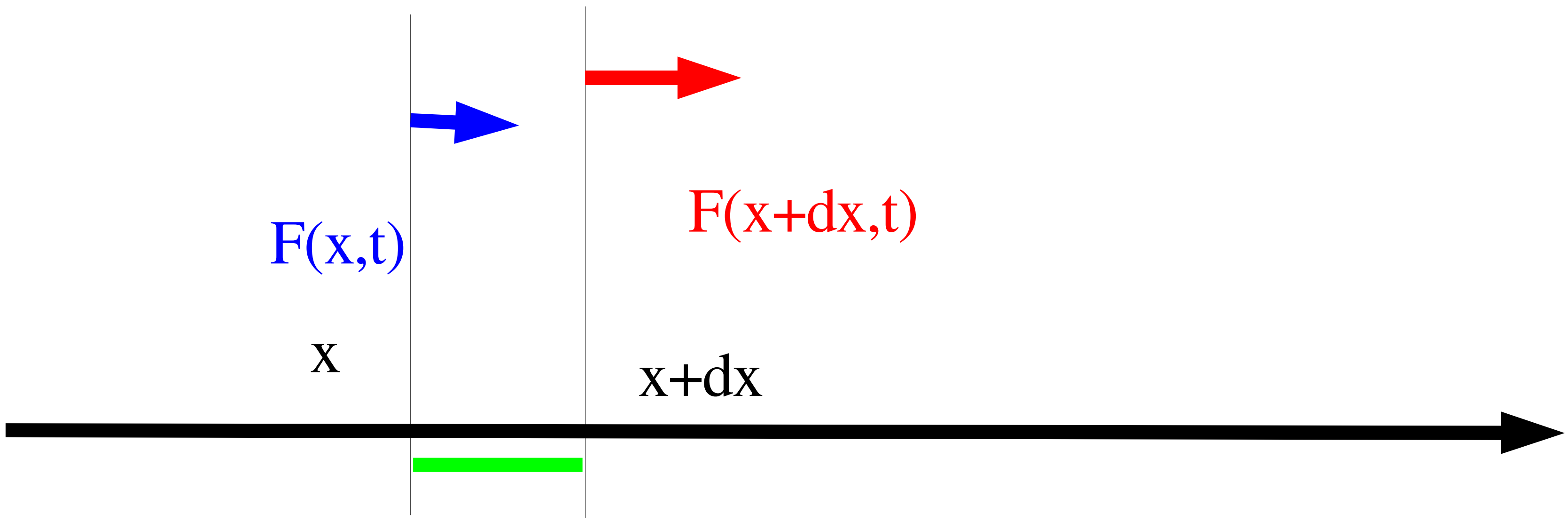
Moderate skewness $\theta = 0.2$

Maximal skewness $\theta = 2 - \alpha$

3. Mass Flux for Lévy flights

3.a. Evolution of the amount of matter between x and x+dx
3.b. Flux through x
3.c. A new expression for the Riemann-Liouville
3.d. Flux and fractional derivative
3.e. Equation

3.a. Evolution of the amount of matter between x and x+dx



C(x,t)dx: matter between x and x+dx at instant t

$$\boxed{\partial_t C(x,t) = \partial_x F(x,t)}$$

3.b. Flux through x

Probability for a walker to cross x to the left during dt:

$$\int_0^{+\infty} C(x+y,t) \int_{-\infty}^{-y} \varphi_l(z)\,dz\, \psi_{\tau_0}(0)\,dy\,dt$$

$$\frac{1}{l}\, p_\alpha^\theta(-y/l)$$

$$\frac{1}{\tau_0}$$

to the right

$$\int_0^{+\infty} C(x-y,t) \int_{y}^{+\infty} \varphi_l(z)\,dz\, \psi_{\tau_0}(0)\,dy\,dt$$

Probability of jumping at least -y to the left

$$F_\alpha^{g,\theta}(-y/l) = \int_{-\infty}^{-y/l} p_\alpha^\theta(z)\,dz$$

$$F_\alpha^{d,\theta} = \int_{y/l}^{+\infty} p_\alpha^\theta(z)\,dz$$

3.b. Net Flux through x:

$$\int_0^{+\infty} F_\alpha^{g,\theta}(-y)\,dy = \int_0^{+\infty} F_\alpha^{d,\theta}(y)\,dy$$

cancel each other

-K times

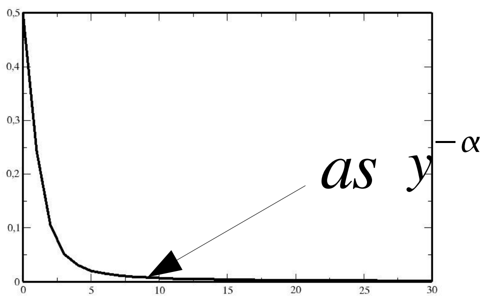
$$\left[\underline{\int_0^{+\infty} C(x+y,t)\frac{F_\alpha^{g,\theta}(-y/l)}{l^\alpha}dy} - \int_0^{+\infty} C(x-y,t)\frac{F_\alpha^{d,\theta}(y/l)}{l^\alpha}dy\right]$$

→ *a right-sided Riemann-Liouville derivative or order* $\alpha - 1$

for $l \to 0$,

+ *a local fractional derivative*

3.c. 1 a new expression for Riemann-Liouville derivatives:



$$\boxed{\int_0^{+\infty} F(y)\,dy = 0} \qquad F = F_1 + F_2$$

F_1 integrable, $\quad F_1(y)\,y^{\alpha+1} \to 0$ as $y \to \infty$

$$\boxed{F_2 = y^{-\alpha} \quad \text{for y>1, 0 for y<1}}$$

e.g: $\quad F_\alpha^{g,\theta}(-.) - \int_0^{+\infty} F_\alpha^{g,\theta}(-y)\,dy\,\chi_{[0,1]}(.)$

$$\boxed{lim_{l\to 0} \int_0^{+\infty} f(x+y)\frac{\boxed{F(y/l)}}{l^\alpha}dy}$$

=

left Inverse of

$$\frac{1}{2\cos(\alpha\pi/2)} \quad \text{times} \quad I_{x,+\infty}^{\alpha-1} f(x) = \frac{1}{\Gamma(\alpha-1)} \int_x^{+\infty} (y-x)^{\alpha-2} f(y)\,dy$$

hence, the above is the Riemann-Liouville (or Marchaud) derivative

$$\frac{1}{2\cos(\alpha\pi/2)} \quad \text{times} \quad D_{x,+\infty}^{\alpha-1} f(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^{+\infty} (y-x)^{-\alpha} f(y)\,dy$$

3.d. Net Flux through x:

if C has a derivative (w.r.t. x)

$$\frac{-K}{\sin(\alpha\pi)}\frac{1}{\Gamma(2-\alpha)}$$

times

$$\frac{d}{dx}\left(-\sin\left(\frac{\alpha+\theta}{2}\pi\right)\int_x^{+\infty}(y+x)^{1-\alpha}C(y,t)\,dy+\sin\left(\frac{\alpha-\theta}{2}\pi\right)\int_x^{-\infty}(x-y)^{1-\alpha}C(y,\right.$$

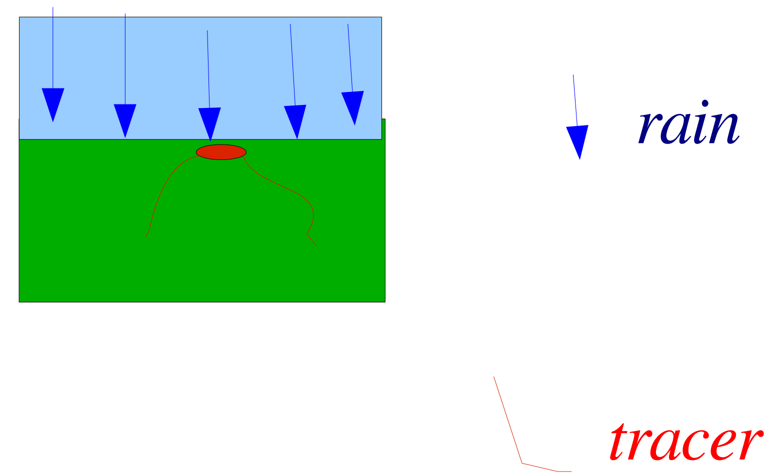
$$\boxed{\partial_t C(x,t)=K\,\nabla_\theta^\alpha C(x,t)}$$

4. Lévy Flights with boundaries

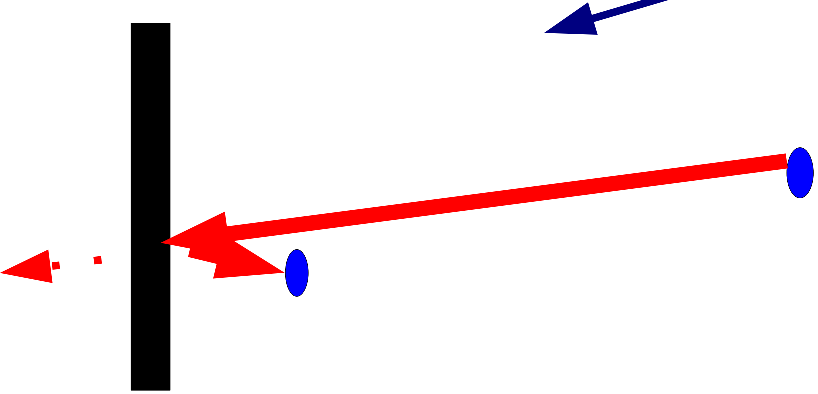
4.a. Motivation: to represent a wall, impermeable for tracer particles

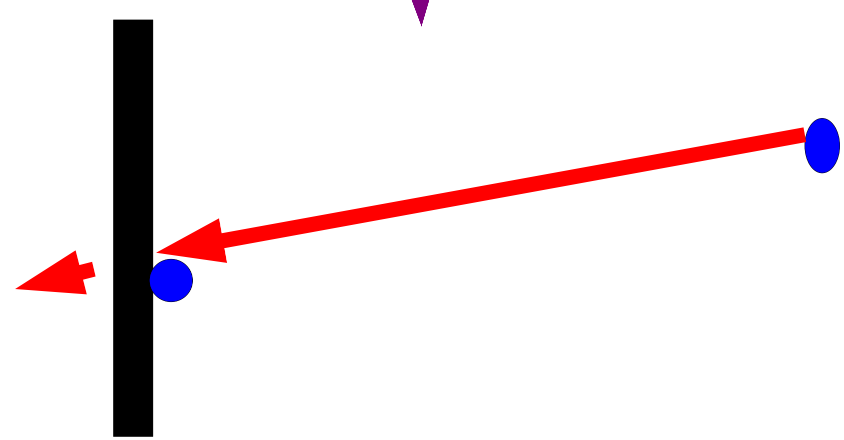
Example 1: an impermeable wall in an experimental device

Example 2: the upper surface of the soil



among several possibilities: "elastic wall" or soft wall"





4.b. Elastic wall

Net flux through x =

$$\frac{K}{\sin(\alpha\pi)\Gamma(2-\alpha)} \quad times \quad \Big[$$

$$\sin\left(\frac{\alpha-\theta}{2}\pi\right)\partial_x \int_{y=0}^{x} (x-y)^{1-\alpha} C(y,t)\,dy \quad +$$

$$\sin\left(\frac{\alpha+\theta}{2}\pi\right)\partial_x \left[\int_{y=0}^{+\infty} (x+y)^{1-\alpha} C(y,t)\,dy + \int_{y=x}^{+\infty} (y-x)^{1-\alpha} C(y,t)\,dy\right] \Big]$$

Elastic wall

$$\partial_t C(x,t) = \frac{-K}{\sin(\alpha\pi)\Gamma(2-\alpha)} \quad \textit{times} \quad \Big[$$

$$\sin\left(\frac{\alpha-\theta}{2}\pi\right)\partial_x^2\left[\int_{y=0}^{x}(x-y)^{1-\alpha}C(y,t)\,dy\right] \quad +$$

$$\sin\left(\frac{\alpha+\theta}{2}\pi\right)\partial_x^2\left[\underline{\int_{y=0}^{+\infty}(x+y)^{1-\alpha}C(y,t)\,dy}+\int_{y=x}^{+\infty}(y-x)^{1-\alpha}C(y,t)\,dy\right] \quad \Big]$$

quite different from

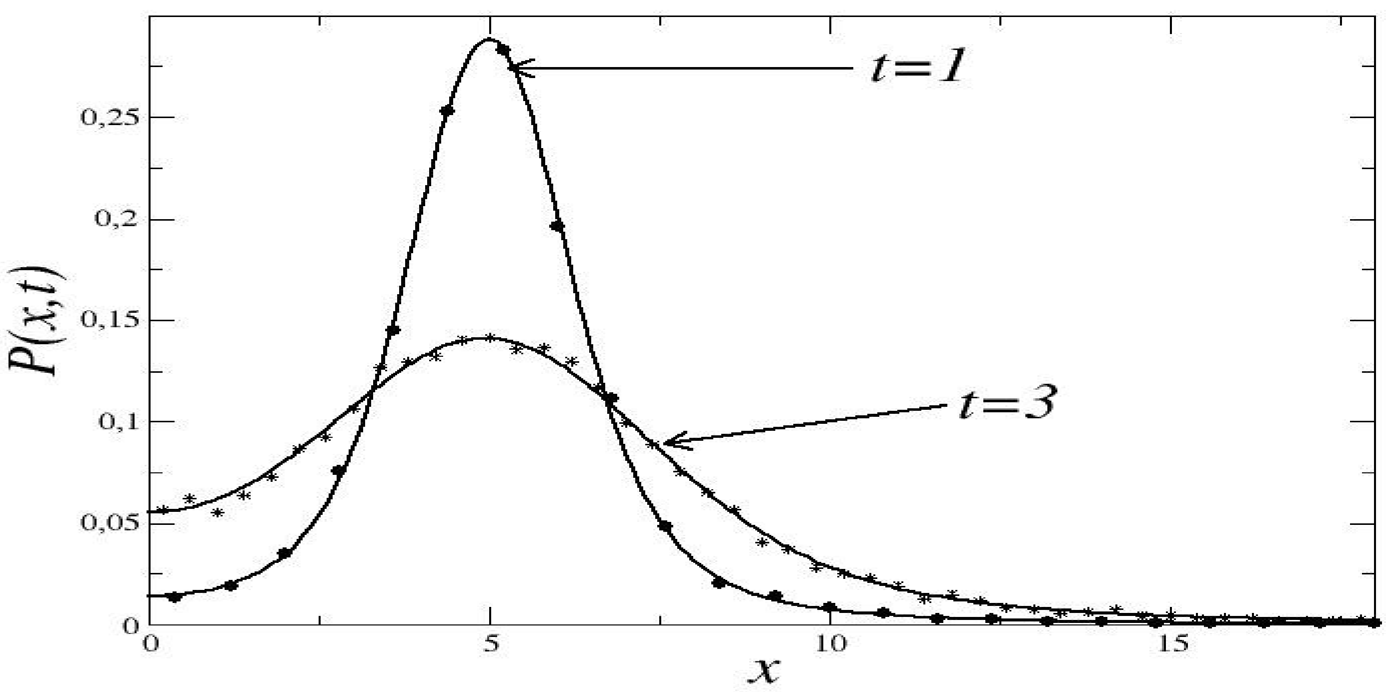
$$\partial_t C(x,t) = \frac{-K}{\sin(\alpha\pi)}\left(\sin(\pi/2(\alpha-\theta))D^{\alpha}_{-\infty,x} + \sin(\pi/2(\alpha+\theta))D^{\alpha}_{x,\infty}\right)$$

with $C(y,t)=0$ for $y<0$

Symmetric case $\theta=0$

$$\partial_t C(x,t)=\frac{-K}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)}\partial_x^2\int_{y=0}^{+\infty}\left(|x-y|^{1-\alpha}+(x+y)^{1-\alpha}\right)C(y,t)\,dy$$

comparison with histograms of Lévy Flights, incorporating ***1 elastic wall***



4.c. Soft wall

Net flux through x = $\dfrac{K}{\sin(\alpha\pi)\Gamma(2-\alpha)}$ *times*

$$\Bigg[\ \sin\left(\frac{\alpha-\theta}{2}\pi\right)\partial_x\int_{y=0}^{x}(x-y)^{1-\alpha}C(y,t)\,dy \quad +$$

$$\sin\left(\frac{\alpha+\theta}{2}\pi\right)\partial_x\int_{y=x}^{+\infty}(y-x)^{1-\alpha}C(y,t)\,dy$$

$$+$$

$$\sin\left(\frac{\alpha+\theta}{2}\pi\right)H(-x)\left[\partial_z\int_{y=z}^{\infty}(y-z)^{1-\alpha}C(y,t)\,dy\right]_{z=0} \quad +$$

$$Cte\, lim_{l\to 0}\left(\frac{C(l,t)-C(0,t)}{l^{\alpha-1}}\right)\ \Bigg]$$

soft wall

$$\partial_t C(x,t) = \frac{-K}{\sin(\alpha\pi)\,\Gamma(2-\alpha)} \quad \textit{times} \quad \Big[$$

$$\sin\left(\frac{\alpha-\theta}{2}\pi\right)\partial_x^2 \int_{y=0}^{x} (x-y)^{1-\alpha} C(y,t)\,dy \quad +$$

$$\sin\left(\frac{\alpha+\theta}{2}\pi\right)\partial_x^2 \int_{y=x}^{+\infty} (y-x)^{1-\alpha} C(y,t)\,dy \quad +$$

as without the wall, with C(y,t)=0 for y<0

$$\sin\left(\frac{\alpha+\theta}{2}\pi\right)\left[\partial_z \int_{y=z}^{+\infty} (y-z)^{1-\alpha} C(y,t)\,dy\right]_{z=0} \delta(x) \quad +$$

$$Cte\, lim_{l\to 0}\left(\frac{C(l,t)-C(0,t)}{l^{\alpha-1}}\right)\delta(x) \quad \Big]$$

for x=0 only

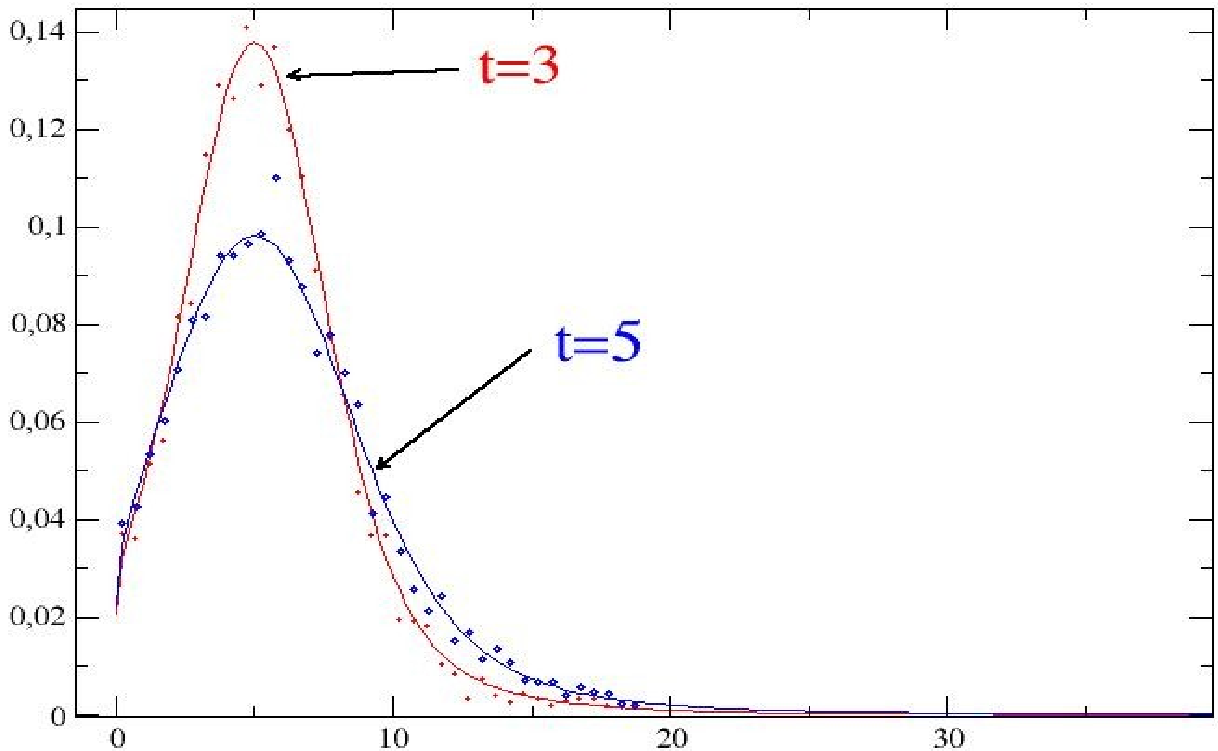
Soft wall, numerical simulation of

$$\partial_t C(x,t) = K \nabla_\theta^\alpha C(x,t)$$

+

$$\delta(x)\left(K_1 D H^{\alpha-1} C(0,t) + K_2 D_{0,+\infty}^{\alpha-1} C\right)$$

compared with histograms of Lévy Flights including a soft wall



in the symmetric case

$$\theta = 0$$

Conclusions

. Fractional equations are widely used for dispersion in heterogeneous media,

. Fractional derivatives are non local operators, inserting boundary conditions sometimes modifies the model. Consequences may be visible far from the boundary

. A novel expression of Riemann-Liouville derivative yields that net mass flux combines skewed Riesz-Feller derivative and local fractional derivative

. Adapts to problems with boundary conditions