Commande - Robotique
Ordres Non Entiers

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FRACTIONAL INTEGRATION NUMERICAL ALGORITHM BASED ON A RECURSIVE DISTRIBUTION OF POLES AND ZEROS: APPLICATION TO IGNITION PREDICTION OF REACTIVE MATERIALS

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Outline

Fractional Integration Numerical Algorithm Based on a Recursive Distribution of Poles and Zeros: Application to Ignition Prediction of Reactive Materials

1 – Ignition of reactive material

2 – Fractional integration for ignition prediction

3 – Description of the numerical method for fractional integration

4 – Results

5 – How to choose synthesis parameters

6 – Conclusion

Ignition of reactive material

Physical phenomenon description

Under specific dynamical solicitations (for instance impacts with speed close to 10 to 100 m/s), deformation mechanisms of some materials lead to local temperature increase (process are adiabatic given the range of time considered).

For the reactive material considered in this study, (Arthenius kinetic is here considered) an small temperature increase leads to a large speed reaction increase. Moreover, these material are very exothermic.

Thus, some critic conditions exist where mechanical dissipation can lead to material ignition.

For safety purpose, models are required to predict the ignition of such materials.

Ignition of reactive material

A numerical model for ignition prediction of reactive material

In 2001, [Browning, 1995], Browning, 2001], the following model was proposed to predict the ignition of reactive materials:

$$I = \int_0^t (t-\tau)^{-n} \varphi(\tau) d\tau < \Omega$$

where function $\varphi(t)$ is given by $\varphi(t) = \left(\frac{\sigma_A(t)}{E}\right)^{\frac{2(n-1)}{3}} \dot{\mu}(t)$

in which $\sigma_A(t)$, E , $\dot{\mu}(t)$ denote respectively the macroscopic pressure, the young modulus and the shear strain rate.

n **is a fractional number close to 0.77 and depending on reactive material**

Ω **denotes the maximal level of integral I that ensure material non ignition**

Ignition of reactive material

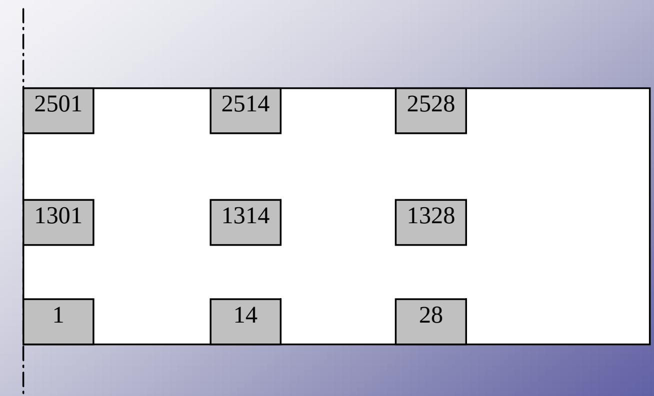
Ignition condition and fractional integration

According to the previous model, there is no ignition of the reactive material if the fractional integral

$$I = \int_0^t (t-\tau)^{-(1-\gamma)} \varphi(\tau) d\tau < \Omega$$

of order $\gamma = 1 - n$ of function $\varphi(t)$ is less then Ω.

To predict the ignition of the material, function $\varphi(t)$ is computed with a finite element software (Abaqus Explicit) at a large number of node, and integral I must be computed at each node.

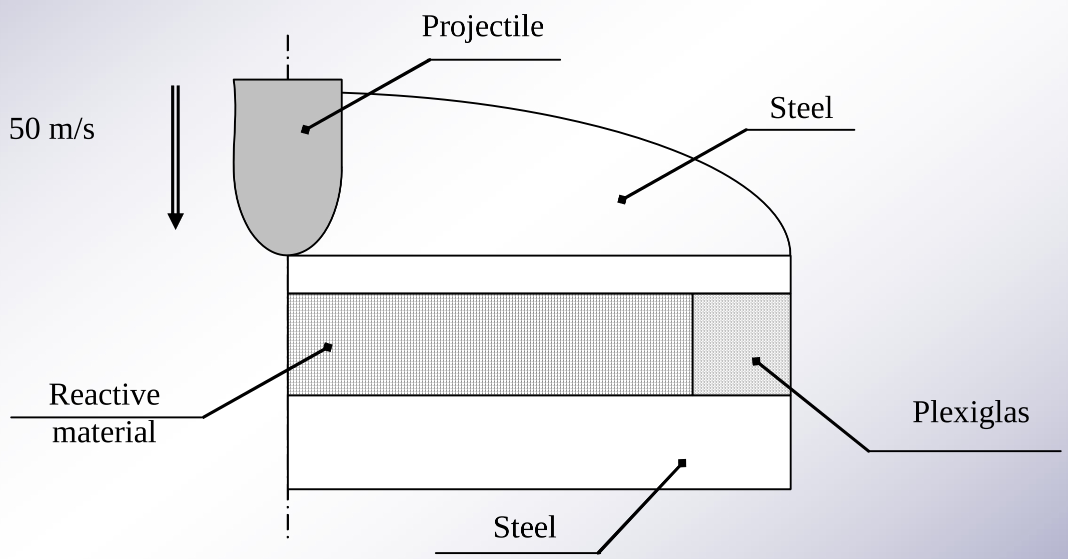


A numerical algorithm for fractional integration is thus required

Ignition of reactive material

A validation test bench

In order to validate the model previously given and the numerical method for fractional integration, a test bench was designed by the CEA society.



A reactive material confined between two steel plate and a plexiglas washer is hit by a projectile whose velocity is close to 50 m/s

Ignition of reactive material

Validation method

Experiment

Impact with a velocity V_0 of the test bench

↓

Ignition : Yes or No ?

Numerical methods

Computation of function $\varphi(t)$ at each node after un impact

↓

Computation of the fractional integral of $\varphi(t)$ at each node

↓

Ignition : Yes or No

Comparison

Fractional integration for ignition prediction

Fractional integration numerical method specifications

Numerical algorithm for fractional integration :

- must allows (quasi) real-time computation (a low number of operation at each node),

- must not require a storage of a large number of sample of function $\varphi(t)$ (given the large number of node, it is impossible to store all the function $\varphi(t)$ history to compute the fractional integral by post computation),

- must allow variable sampling period (a variable step method is used to produce functions $\varphi(t)$),

- must be implemented easily and the user must be guided to select implementation parameters,

- approximation error must be predicted.

Description of the numerical method for fractional integration

Fractional integration: a filtering problem

$$I^{\gamma}(t) = \mathcal{L}^{-1}\left\{\frac{1}{p^{\gamma}}\right\}$$

$f(t)$

$$I^{\gamma}(p) = \frac{1}{p^{\gamma}}$$

$$\frac{1}{\Gamma(\gamma)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\gamma}}\,d\tau$$

Description of the numerical method for fractional integration

Fractional integration and recursivity

Laplace transform

$$I^{\gamma}(t)=\frac{\sin(\gamma\pi)}{\pi}\int_{0}^{\infty}\frac{e^{-tx}}{x^{\gamma}}dx \qquad n\in\mathbb{R} \qquad 0<\gamma<1 \qquad I^{\gamma}(t)=\mathcal{L}^{-1}\left\{\frac{1}{p^{\gamma}}\right\}$$

$$I^{\gamma}(p)=\frac{\sin(\gamma\pi)}{\pi}\int_{0}^{\infty}\frac{1}{x^{\gamma}(p+x)}dx$$

Variable change

$$dx=e^{-z}dz$$
$$x=e^{-z}$$

$$I^{\gamma}(p)=\frac{\sin(\gamma\pi)}{\pi}\int_{-\infty}^{\infty}\frac{e^{-z}}{e^{-\gamma z}\left(p+e^{-z}\right)}dz=\frac{\sin(\gamma\pi)}{\pi}\int_{-\infty}^{\infty}\frac{e^{\gamma z}}{\left(\frac{p}{e^{-z}}+1\right)}dz$$

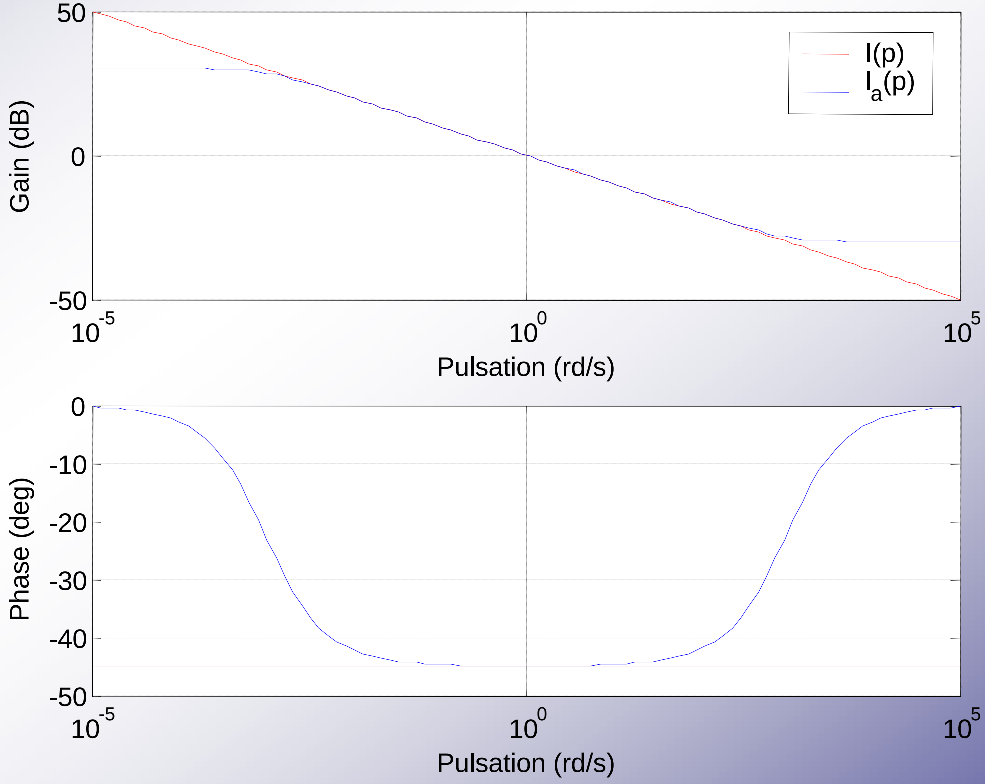
Integral discretisation

$$I^{\gamma}(p)=\frac{\sin(\gamma\pi)}{\pi}\sum_{k=-\infty}^{\infty}\frac{e^{\gamma k\Delta z}}{\frac{p}{e^{-k\Delta z}}+1}=\frac{\sin(\gamma\pi)}{\pi}\sum_{k=-\infty}^{\infty}\frac{e^{(\gamma-1)k\Delta z}}{p+e^{-k\Delta z}}$$

Description of the numerical method for fractional integration

First step : frequency band truncation

$$I^{\gamma}(p) \approx I_a^{\gamma}(p) = C_0 \left(\frac{1 + \frac{p}{\omega_h}}{1 + \frac{p}{\omega_b}} \right)^{\gamma}$$



Description of the numerical method for fractional integration

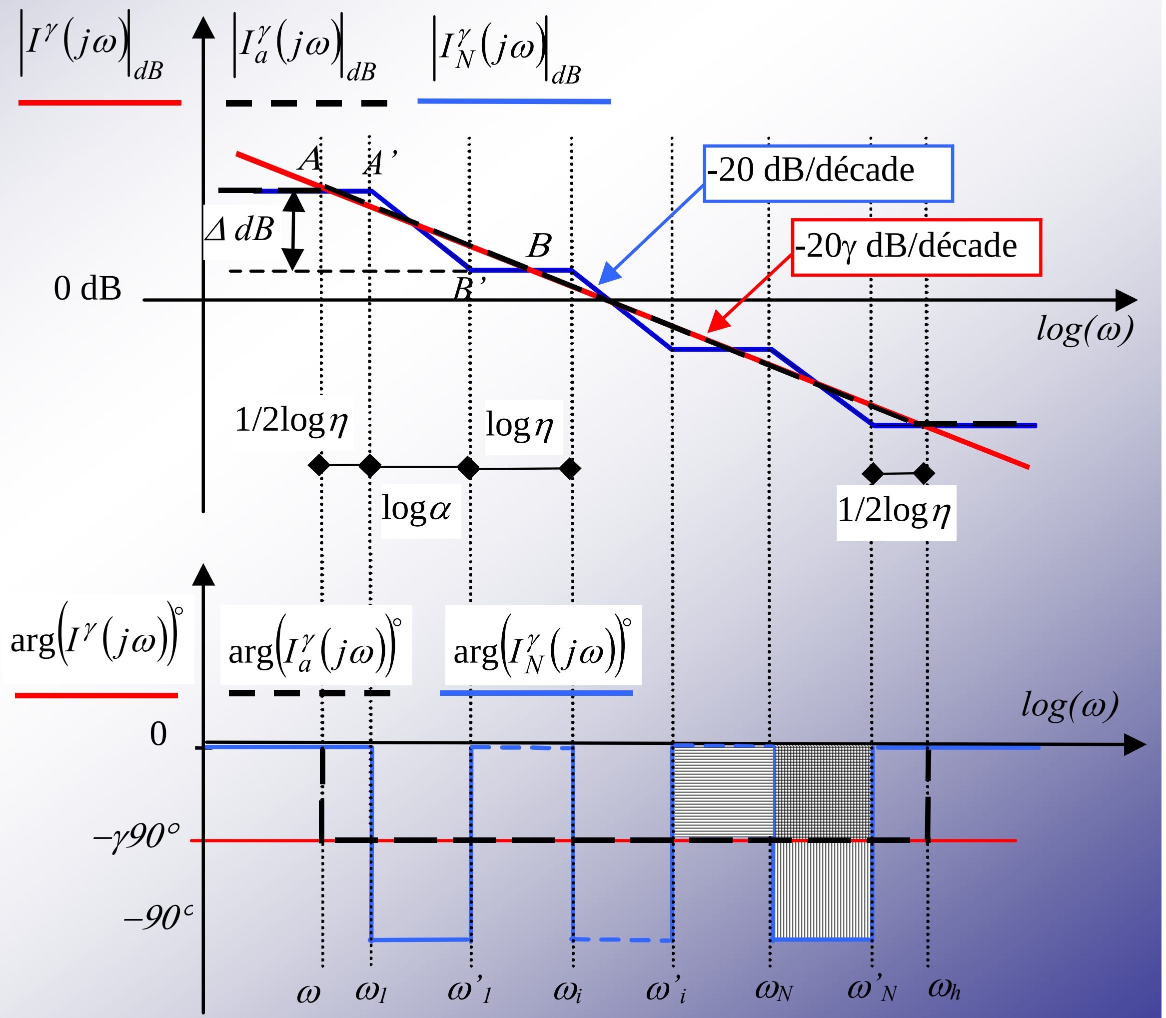
Second step : frequency discretisation

$$I_a^\gamma(p) = \left(\frac{1+\frac{p}{\omega_h}}{1+\frac{p}{\omega_b}}\right)^\gamma \approx I_N^\gamma(p) = \frac{\prod_{k=1}^{N}\left(1+\frac{p}{\omega'_k}\right)}{\prod_{k=1}^{N}\left(1+\frac{p}{\omega_k}\right)}$$

$$\omega_1 = \omega_b\sqrt{\eta}$$

$$\omega'_{i+1} = \alpha\eta\omega'_i \qquad \omega_{i+1} = \alpha\eta\omega_i$$

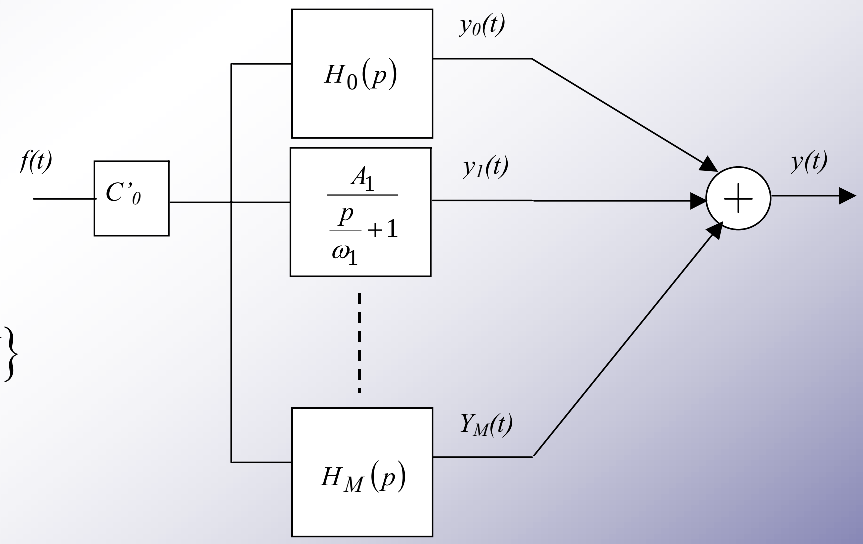
$$\alpha\eta = 10^{\frac{\log\left(\frac{\omega_h}{\omega_b}\right)}{N}}$$



Description of the numerical method for fractional integration

Third step : fraction expansion

$$I_{a_1}^{\gamma}(p) = C'_0 \frac{\prod_{k=1}^{N}\left(\frac{p}{\omega'_k}+1\right)}{\prod_{k=1}^{N}\left(\frac{p}{\omega_k}+1\right)} = C'_0\left(A_0 + \sum_{k=1}^{N}\frac{A_k}{\frac{p}{\omega_k}+1}\right)$$



$$A_0 = 1 \qquad A_i = \frac{\prod_{k=1}^{N}\left(\frac{-\omega_i}{\omega'_k}+1\right)}{\prod_{\substack{k=1\\k\neq i}}^{N}\left(\frac{-\omega_i}{\omega_k}+1\right)} \qquad i \in \{1 \cdots N\}$$

Interest :

- **a lower error propagation**
- **a smaller memory**

Description of the numerical method for fractional integration

Fourth step : time discretisation

The Euler approximation is used for time discretisation

$$\frac{d}{dt} \approx \frac{\left(1-z^{-1}\right)}{h} \quad \Longrightarrow \quad p \approx \frac{\left(1-z^{-1}\right)}{h}$$

$$y_k(t_i) = \frac{y_k(t_{i-1})}{1+(t_i - t_{i-1})\omega_k} + \frac{C'_0\, A_k \omega_k u(t_i)}{1+(t_i - t_{i-1})\omega_k} \qquad i \geq 1 \qquad y_k(0) = 0 \qquad t_0 = 0$$

$$y(t_i) = A_0 C'_0\, u(t_i) + \sum_{k=1}^{M} y_k(t_i)$$

Others approximation formulas where also used

- Tustin $\frac{d}{dt} = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}$ **- Al Alaoui** $\frac{d}{dt} = \frac{8}{7h}\frac{1-z^{-1}}{1+z^{-1}/7}$

A small accuracy improvement however, a larger memory size is required

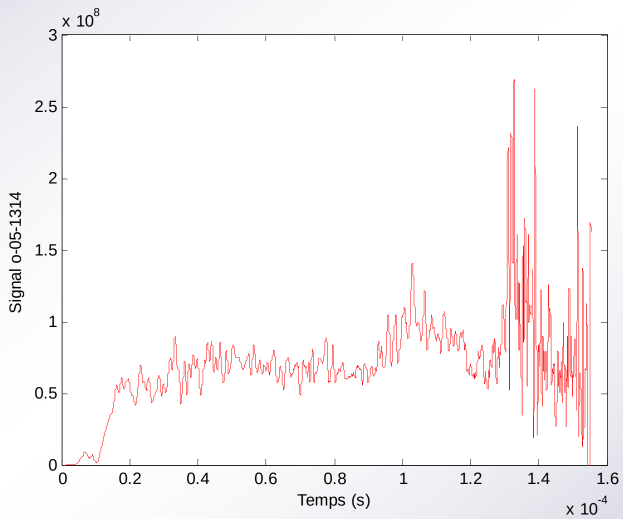
Results

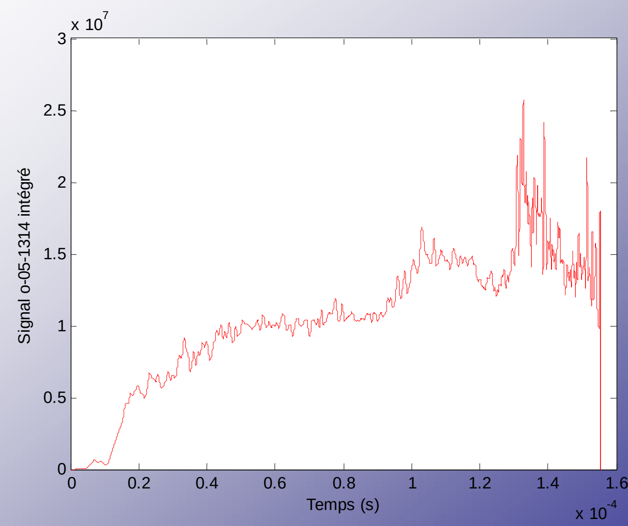
A reference signal computation

$$u(t) = \sum_{k=0}^{M} u_k \left(H(t - t_k) - H(t - t_{k+1}) \right)$$

$$I^{\gamma}\{u(t)\} = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{\sum_{k=0}^{M} u_k \left(H(\tau - t_k) - H(\tau - t_{k+1}) \right)}{(t-\tau)^{1-\gamma}} d\tau = \sum_{k=0}^{M} u_k \frac{1}{\Gamma(\gamma)} \int_0^t \frac{\left(H(\tau - t_k) - H(\tau - t_{k+1}) \right)}{(t-\tau)^{1-\gamma}} d\tau$$

$$I^{\gamma}\{u(t)\} = \sum_{k=0}^{M} u_k \frac{1}{\Gamma(\gamma+1)} \left((t - t_k)^{\gamma} H(\tau - t_k) - (t - t_{k+1})^{\gamma} H(\tau - t_{k+1}) \right)$$





Results

Application of the proposed numerical algorithm

$$I^{\gamma}(p) \approx I^{\gamma}_{a_1}(p) = C_0 \left(\frac{1+\frac{p}{\omega_h}}{1+\frac{p}{\omega_b}} \right)^{\gamma} \approx I^{\gamma}_{N_1}(p) = C'_0 \frac{\prod_{k=1}^{N}\left(\frac{p}{\omega'_k}+1\right)}{\prod_{k=1}^{N}\left(\frac{p}{\omega_k}+1\right)}$$

$$\omega_b = 1{,}25 rd/s$$

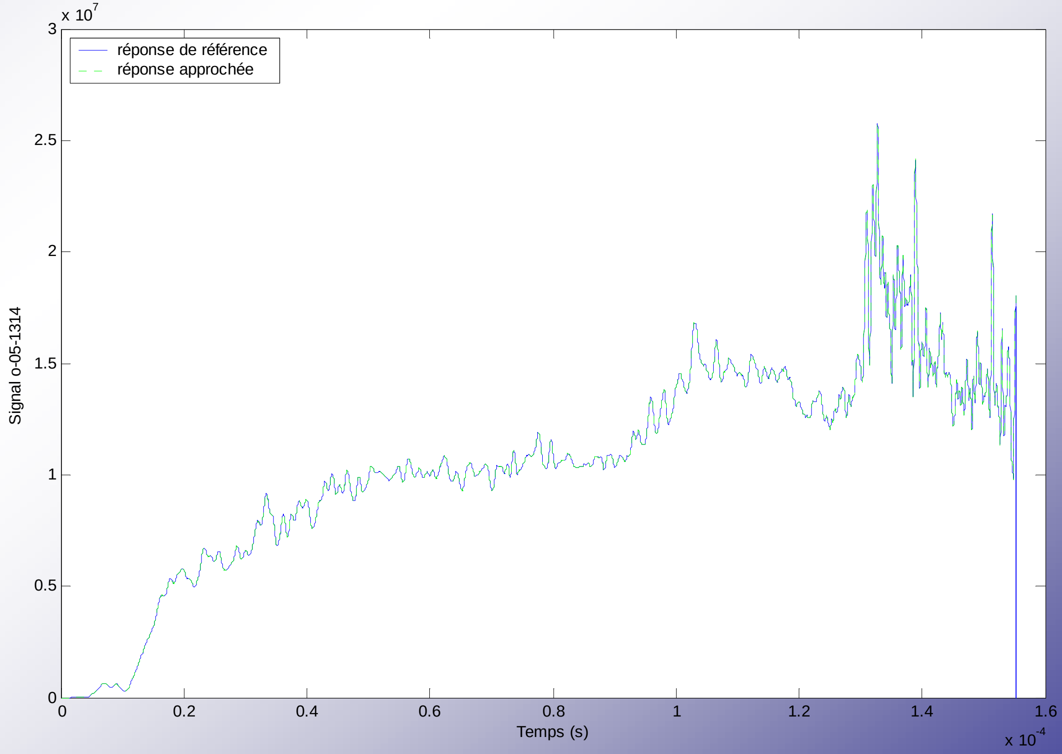
$$\omega_h = 3{,}14.10^{9} rd/s$$

$$\alpha\eta = 10^{\frac{\log\left(\frac{\omega_h}{\omega_b}\right)}{N}} = 4.23$$

$$\alpha = 10^{\gamma \log(\alpha\eta)} = 1.33$$

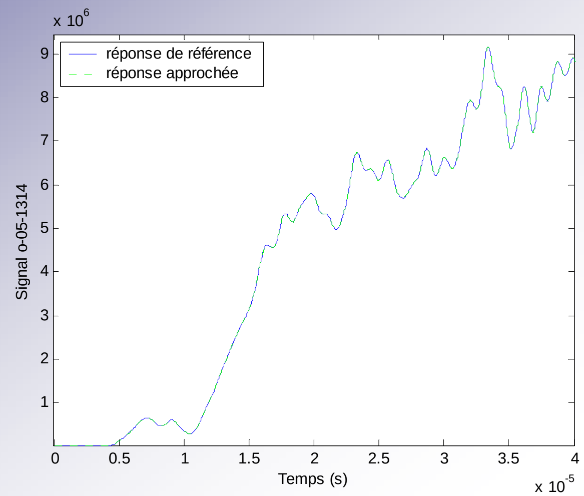
$$\omega'_1 = \alpha\omega_1 \qquad \omega_1 = \omega_b\sqrt{\eta}$$

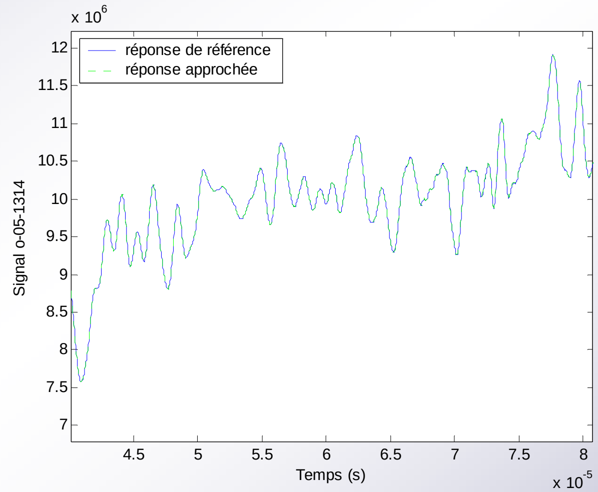
$$\omega'_{k+1} = \alpha\eta\omega'_k \qquad \omega_{k+1} = \alpha\eta\omega_k$$

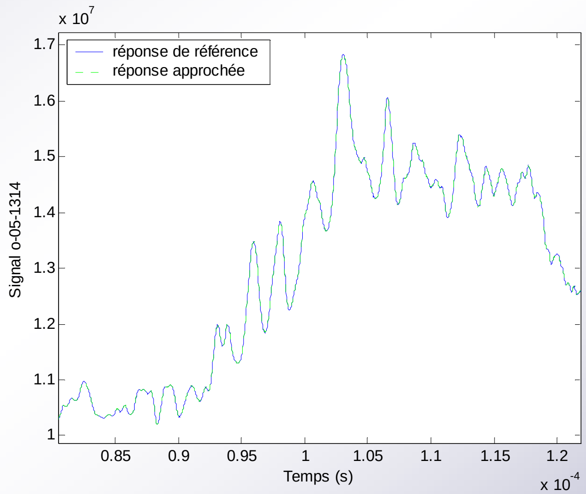


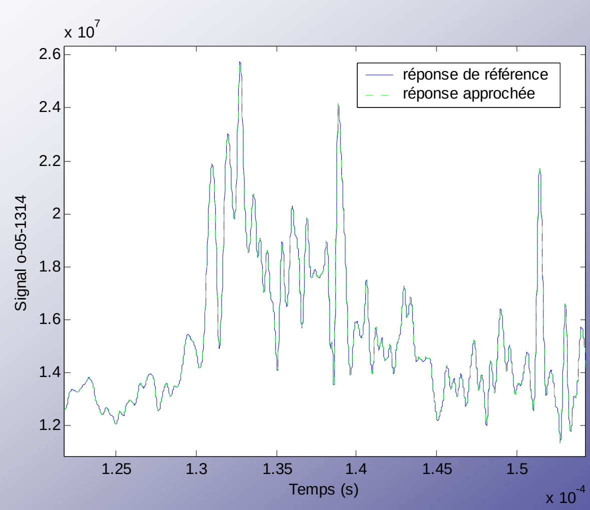
Results

Application of the proposed numerical algorithm





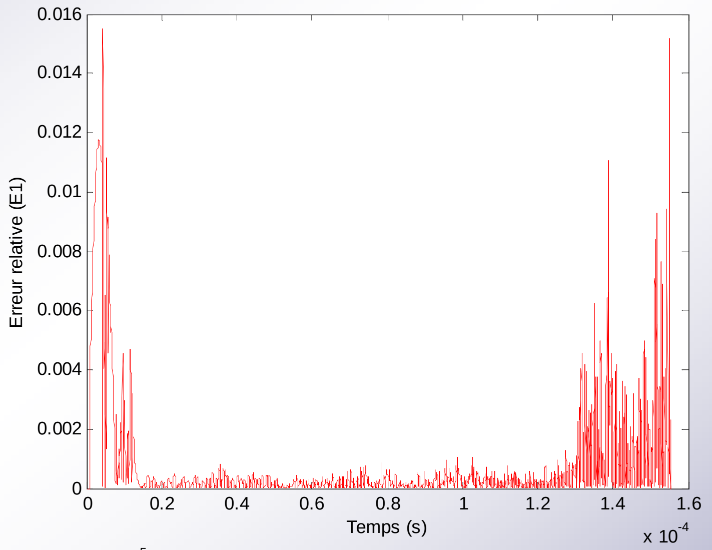




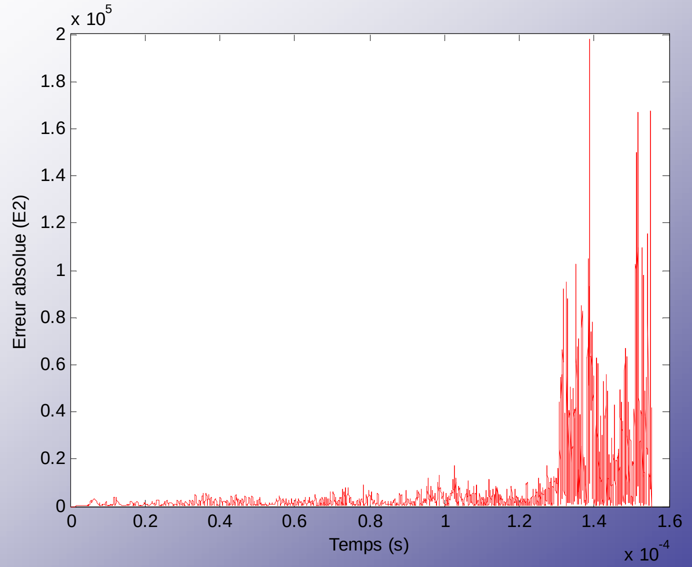
Results

Error computation

$$E_1(t_i) = \frac{\left| y_{ref}(t_i) - y(t_i) \right|}{y_{ref}(t_i)}$$



$$E_2(t_i) = \left| y_{ref}(t_i) - y(t_i) \right|$$



How to chose synthesis parameters ?

Through rule of thumb

$$I^{\gamma}(p) \approx I_{a_1}^{\gamma}(p) = C_0 \left(\frac{1+\frac{p}{\omega_h}}{1+\frac{p}{\omega_b}} \right)^{\gamma} \approx I_{N_1}^{\gamma}(p) = C'_0 \frac{\prod_{k=1}^{N} \left(\frac{p}{\omega'_k} + 1 \right)}{\prod_{k=1}^{N} \left(\frac{p}{\omega_k} + 1 \right)}$$

If T_{fin} denotes the measure time duration, parameter ω_b must be choosen equal to :

$$\omega_b = \frac{1}{1e3} \frac{2\pi}{T_{fin}}$$

If ΔTmin denotes the minimal sampling time, parameter ω_h must be choosen equal to :

$$\omega_h = \frac{2\pi}{\Delta T_{\min}}$$

Parameter N must meet (see also work by Guglielemi):

$$N \approx 1.5 \log\left(\frac{\omega_h}{\omega_h} \right)$$

How to chose synthesis parameters ?

Through error prediction

Goal : choice of ω_b and ω_h only with the knowledge of T_{fin} and $\max(T_{k+1} - T_k)$

$$u(t) = \sum_{k=1}^{M} a_k \left(H(t - T_k) - H(t - T_{k+1})\right) \quad \Rightarrow \quad Y(p) = \frac{1}{p^\gamma} \sum_{k=1}^{M} \frac{a_k}{p} \left(e^{-T_k p} - e^{-T_{k+1} p}\right)$$

$$y_{fen}(t) = y(t)H(t) - y(t)H(t - T_{fin}) \qquad Y_{fen}(p) = Y(p)\left(1 - e^{-T_{fin} p}\right)$$

$$Y_{fen}(p) = \frac{1}{p^\gamma}\left(\sum_{k=1}^{M} \frac{a_k}{p}\left(e^{-T_k p} - e^{-T_{k+1} p}\right)\right)\left(1 - e^{-T_{fin} p}\right)$$

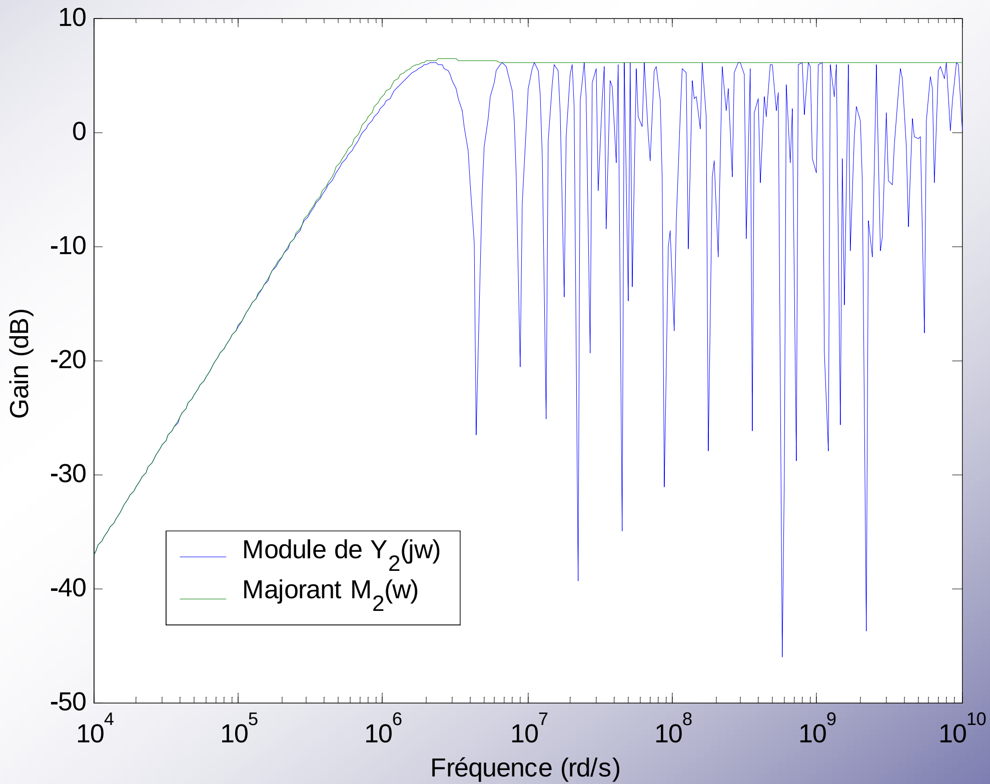
$$Y_1(p) = \sum_{k=1}^{M} \frac{a_k}{p}\left(e^{-T_k p} - e^{-T_{k+1} p}\right) \qquad Y_2(p) = \left(1 - e^{-T_{fin} p}\right)$$

$$\left|Y_{fen}(p)\right|_{p=j\omega} = \left|\frac{1}{(j\omega)^n}\right| \left|Y_1(j\omega)\right| \left|Y_2(j\omega)\right| = \left|\frac{1}{\omega^n}\right| \left|Y_1(j\omega)\right| \left|Y_2(j\omega)\right|$$

How to chose synthesis parameters ?

Through error prediction: study of $\left|Y_2(j\omega)\right|$

$$Y_2(p) = \left(1 - e^{-T_{fin}p}\right)$$



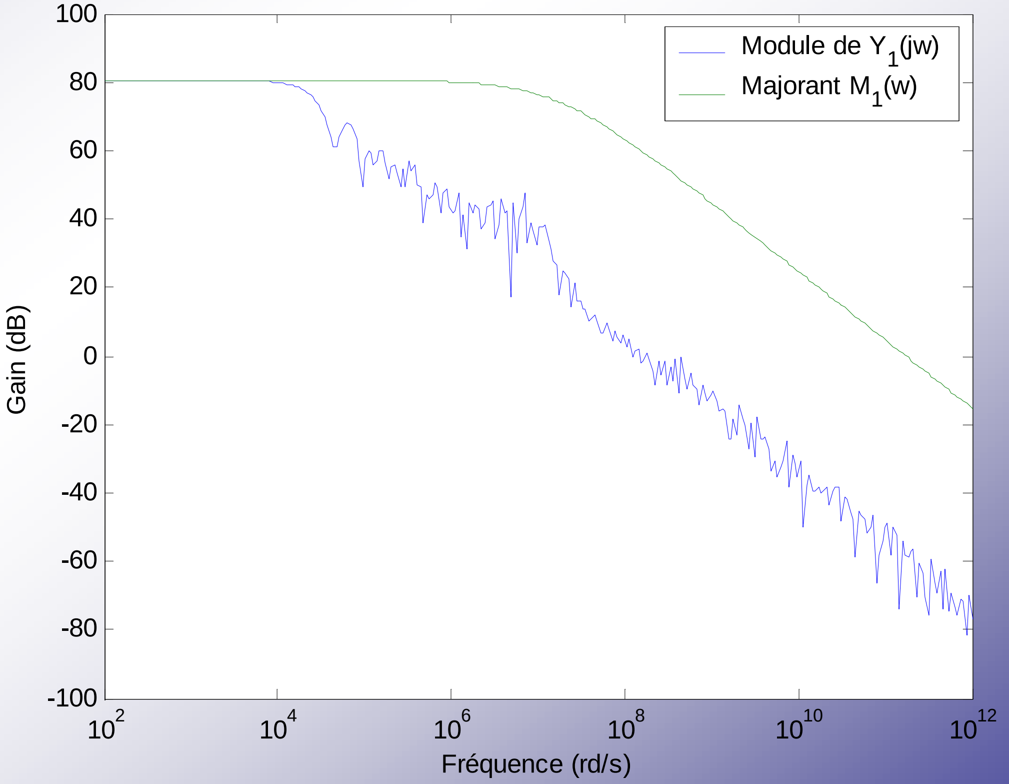
$$\left|Y_2(j\omega)\right| \le \frac{2\dfrac{\omega}{\omega_{0_2}}\sqrt{1+\left(\dfrac{\omega}{\omega_{0_2}}\right)^2}}{\sqrt{\left(1-\left(\dfrac{\omega}{\omega_{0_2}}\right)^2\right)^2 + 4\xi\left(\dfrac{\omega}{\omega_{0_2}}\right)^2}} = M_2(\omega)$$

How to chose synthesis parameters ?

Through error prediction: study of $\left|Y_1(j\omega)\right|$

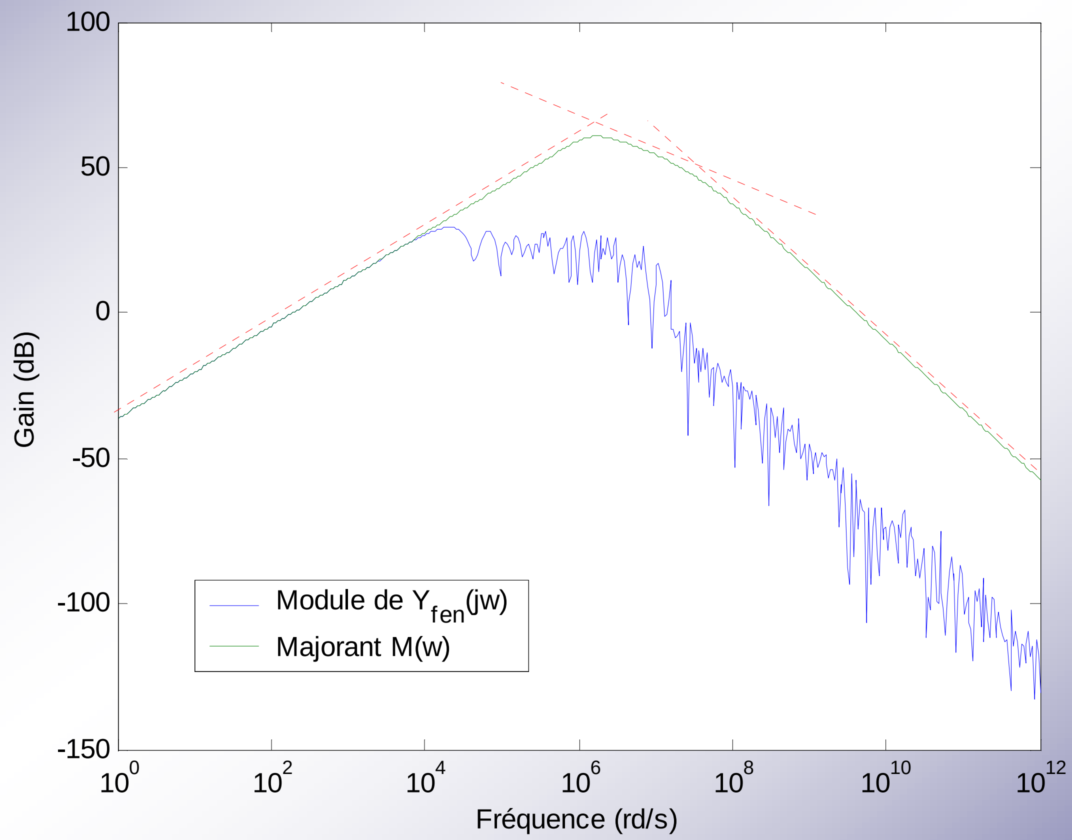
$$Y_1(p) = \sum_{k=1}^{M} \frac{a_k}{p}\left(e^{-T_k p} - e^{-T_{k+1} p}\right)$$

$$\left|Y_1(j\omega)\right| \leq \frac{\sum_{k=1}^{M} a_k \left|T_{k+1} - T_k\right|}{\frac{\omega}{\omega_{0_1}} + 1} = M_1(\omega)$$



How to chose synthesis parameters ?

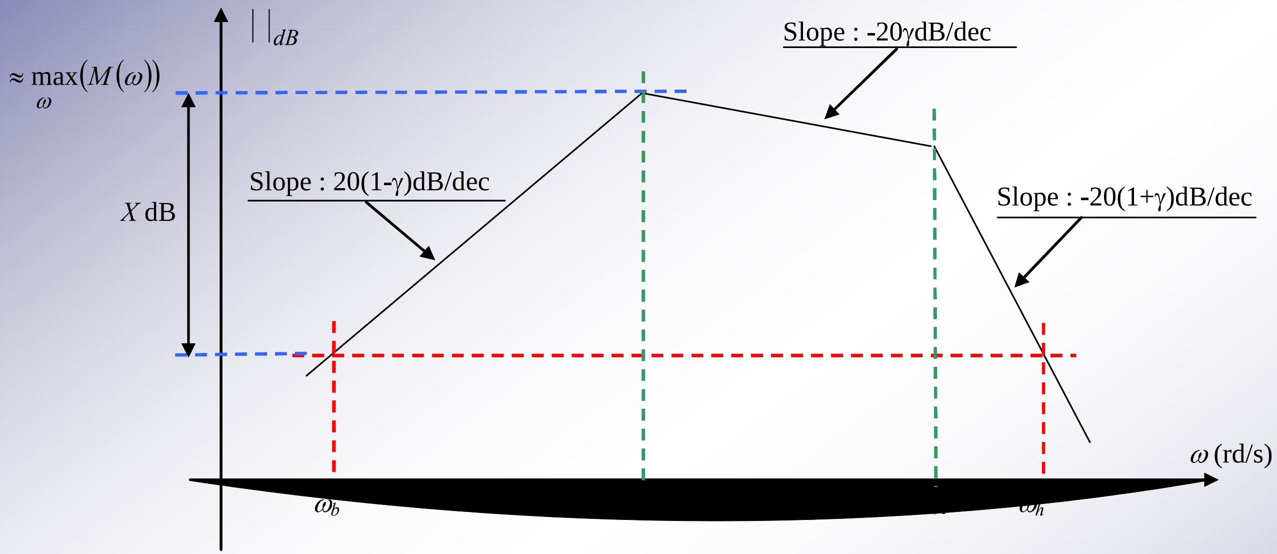
Through error prediction: study of $\left|Y_{fen}(j\omega)\right|$



$$\left|Y_{fen}(j\omega)\right| \leq \frac{1}{\omega^n} M_1(\omega) M_2(\omega) = M(\omega)$$

How to chose synthesis parameters ?

Through error prediction: Choice of ω_b and ω_h



$$\omega_{0_2} = \frac{2}{T_{fin}}$$

$$\omega_{0_1} = \frac{1}{\max(T_{k+1} - T_k)}$$

An idea get a satisfactory approximation of transfer function $1/p^{\gamma}$: compute ω_b and ω_h to define the frequency band with in $M(\omega)$ is greater than $\max_{\omega}(M(\omega)) - X_{dB}$

$$\omega_b = \omega_{0_2} 10^{-\frac{X}{20(1-\gamma)}}$$

$$\omega_h = \omega_{0_1} 10^{\frac{X - 20\gamma \log\left(\frac{\omega_{0_1}}{\omega_{0_2}}\right)}{20(1+\gamma)}}$$

Conclusions

- Fractional integration has been used to predict reactive material ignition

- Among the existing numerical algorithm existing in the literature and tested, algorithm based on a recursive distribution of poles and zeros is the only algorithm that simultaneously :

☻permits real-time computation

☻ requires a storage of a small number of sample of function history (10 to 15 samples are required)

☻ allows a variable sampling period

☻ leads to and easy implementation and the user is guided to select implementation parameters.

Fractional integration for ignition prediction

A comparison of several limited frequency band approximations

$$I^{\gamma}(p) \approx I^{\gamma}_{a_2}(p) = C_0 \left(\frac{1+\dfrac{p}{\omega_h}}{1+\dfrac{p}{\omega_b}} \right)^{\gamma} \frac{1}{1+\dfrac{p}{\omega'_h}}$$

$$I^{\gamma}(p) \approx I^{\gamma}_{a_3}(p) = C_0 \frac{1+\dfrac{p}{\omega'_b}}{\dfrac{p}{\omega'_b}} \left(\frac{1+\dfrac{p}{\omega_h}}{1+\dfrac{p}{\omega_b}} \right)^{\gamma}$$

$$I^{\gamma}(p) \approx I^{\gamma}_{a_4}(p) = \frac{C_0}{p} \left(\frac{1+\dfrac{p}{\omega_b}}{1+\dfrac{p}{\omega_h}} \right)^{1-\gamma}$$

$$I^{\gamma}(p) \approx I^{\gamma}_{a_5}(p) = C_0 \left(\frac{1+\dfrac{p}{\omega_h}}{1+\dfrac{p}{\omega_b}} \right)^{\gamma} \left(\frac{1+\dfrac{p}{\omega_b}}{(1-\gamma)\omega_b\omega_h + \omega_h p + \gamma p^2} \right)$$

No accuracy improvements if the same numerical complexity is imposed (same history storage).

Fractional integration for ignition prediction

A comparison of several time discretisation method

Euler formula $$\frac{d}{dt} = \frac{\left(1 - z^{-1}\right)}{h}$$

Tustin formula $$\frac{d}{dt} = \frac{2}{h}\frac{1 - z^{-1}}{1 + z^{-1}}$$

Al-Alaoui formula $$\frac{d}{dt} = \frac{8}{7h}\frac{1 - z^{-1}}{1 + z^{-1}/7}$$

Low accuracy improvement but a large numerical complexity increase.