بسم الله الرحمن الرحيم

السلام عليكم ورحمة الله وبركاته

SOLUTION OF THE FUNDAMENTAL LINEAR FRACTIONAL ORDER DIFFERENTIAL EQUATION

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OUTLINE

- **Introduction**
- **Relaxation Fractional Order System**
- **Oscillation Fractional order System**
- **Conclusion**

Introduction

Fundamental linear fractional order differential equation

$$(\tau_0)^m \frac{d^m x(t)}{dt^m} + x(t) = e(t) \quad , \qquad \text{for} \quad 0 < m < 2$$

Its corresponding transfer function

$$H(s) = \frac{X(s)}{E(s)} = \frac{1}{[1 + (s\tau_0)^m]} \quad , \qquad \text{for} \quad 0 < m < 2$$

Relaxation Fractional Order System

Relaxation Fractional Order System

$$(\tau_0)^m \frac{d^m x(t)}{dt^m} + x(t) = e(t) \quad , \qquad \text{for} \quad 0 < m < 1$$

Its corresponding transfer function

$$H(s) = \frac{X(s)}{E(s)} = \frac{1}{1+(s\tau_0)^m} \quad , \qquad \text{for} \quad 0 < m < 1$$

The expansion about s = ∞ using long division gives

$$H(s) = \frac{1}{1+(s\tau_0)^m} = \frac{\tau_0^{-m}}{s^m} \sum_{n=0}^{L} \frac{[-(\tau_0^{-m})]^n}{s^{nm}}$$

Relaxation Fractional Order System

$$H(s)=\frac{1}{1+(s\tau_0)^m}=\int_0^\infty \frac{G(\tau)}{1+s\tau}d\tau \quad , \quad \text{for} \quad 0<m<1$$

$$G(\tau)=\frac{1}{2\pi}\left[\frac{\sin(1-m)\pi}{\left\{\cosh\left[m\log\left(\frac{\tau}{\tau_0}\right)\right]\right\}-\cos(1-m)\pi}\right]$$

$$H(s)=\frac{1}{1+(s\tau_0)^m}\cong\sum_{i=1}^{2N-1}\frac{G(\tau_i)}{1+s\tau_i}=\sum_{i=1}^{2N-1}\frac{k_i}{\left(1+\frac{s}{p_i}\right)}$$

Relaxation Fractional Order System

where

$$k_i = G(\tau_i) = \frac{1}{2\pi}\left[\frac{\sin[(1-m)\pi]}{\cosh[m\log(\frac{\tau_i}{\tau_0})] - \cos[(1-m)\pi]}\right]$$

$$p_i = \frac{1}{\tau_i} = (\lambda)^{(i-N)}\frac{1}{\tau_0} \quad , \qquad \text{for } i=1,2,\ldots,2N-1$$

$$p_N = \frac{1}{\tau_0} = \frac{1}{\tau_N}$$

and the parameter λ is the sampling period

Relaxation Fractional Order System

$$H(s) = \frac{1}{1+(s\tau_0)^m} \cong \sum_{i=1}^{2N-1} \frac{G(\tau_i)}{1+s\tau_i} = \sum_{i=1}^{2N-1} \frac{k_i}{\left(1+\frac{s}{p_i}\right)}$$

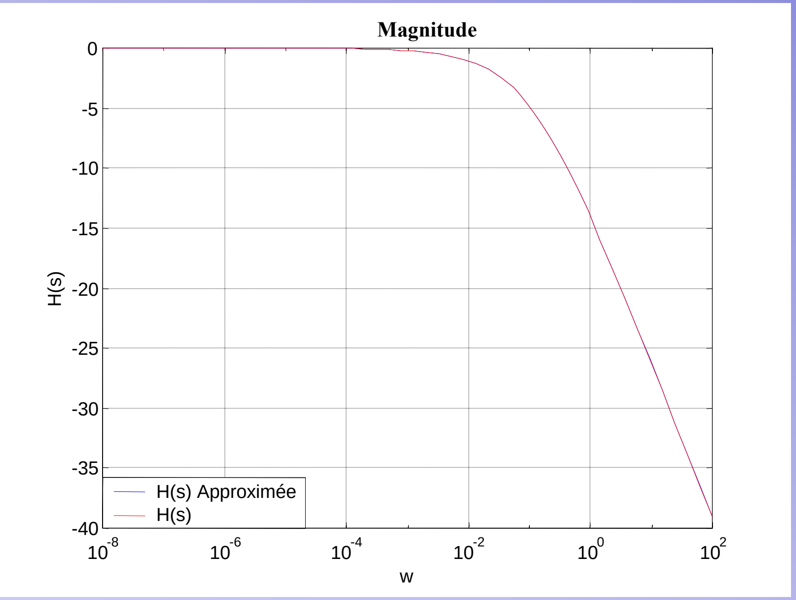
$$H(s) = \frac{1}{1+(s\tau_0)^m} = \frac{\tau_0^{-m}}{s^m} \sum_{n=0}^{L} \frac{[-(\tau_0^{-m})]^n}{s^{nm}}$$

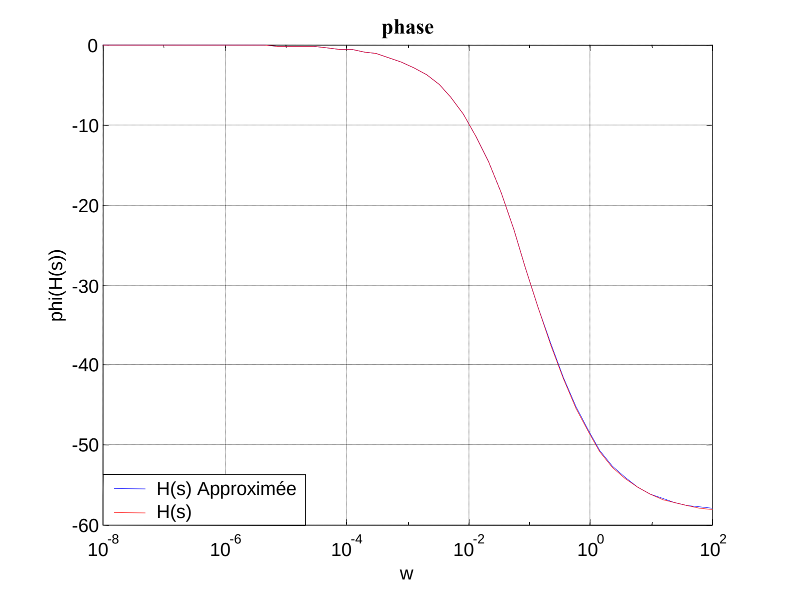
Relaxation Fractional Order System

Numerical example

$$(10)^{0.65}\frac{d^{0.65}y(t)}{dt^{0.65}}+y(t)=e(t)$$

$$H(s)=\frac{1}{1+(10s)^{0.65}}$$

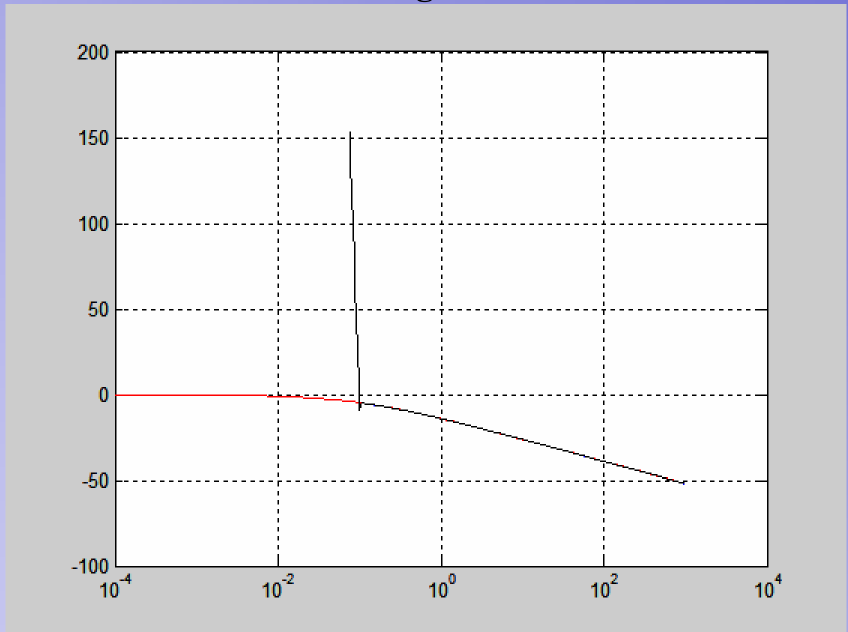




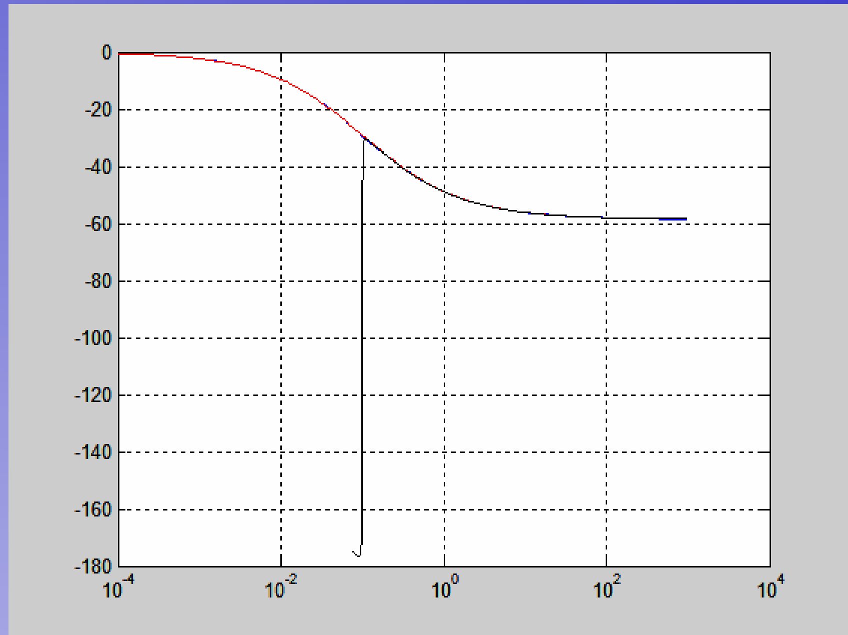
$$H(s)=\frac{1}{1+(s\tau_0)^m}\cong\sum_{i=1}^{19}\frac{G(\tau_i)}{1+s\tau_i}=\sum_{i=1}^{19}\frac{k_i}{\left(1+\frac{s}{p_i}\right)}$$

Relaxation Fractional Order System

Magnitude



Phase



$$H(s) = \frac{1}{1+(s\tau_0)^m} = \frac{\tau_0^{-m}}{s^m}\sum_{n=0}^{100}\frac{[-(\tau_0^{-m})]^n}{s^{nm}}$$

Relaxation Fractional Order System

Impulse response

$$h(t) = \sum_{i=1}^{2N-1} \left(\frac{k_i}{\tau_i} \exp(-t/\tau_i) \right)$$

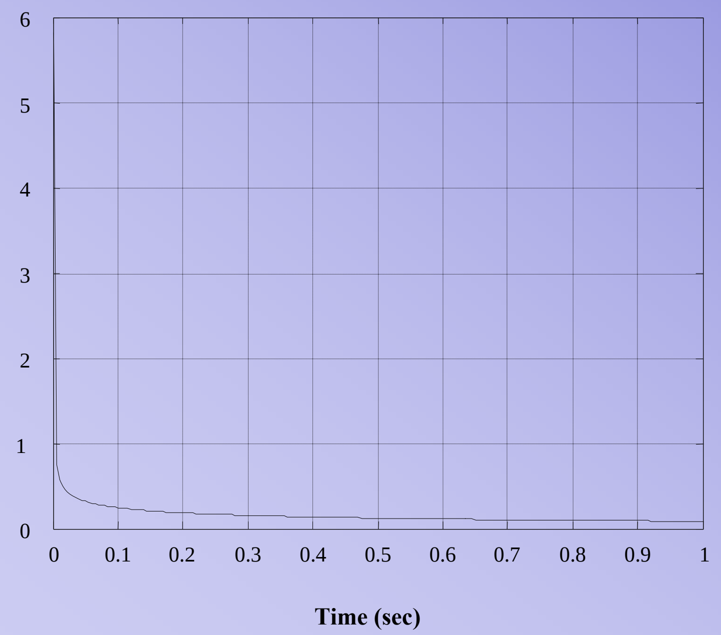
$$h(t) = (\tau_0^{-m}) t^{(m-1)} \sum_{n=0}^{L} \frac{(-(\tau_0^{-m}))^n t^{nm}}{\Gamma(nm+m)}$$

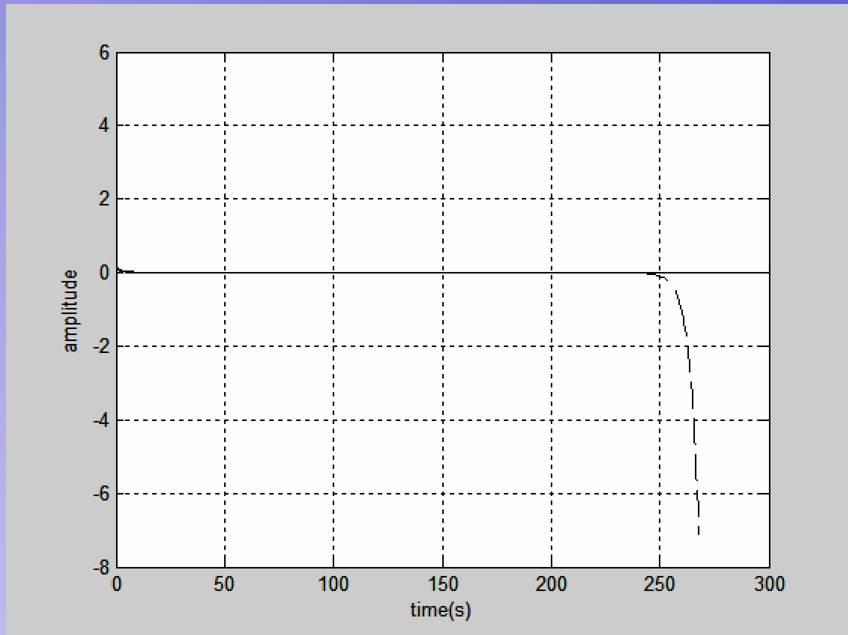
Relaxation Fractional Order System

Numerical example

$$(10)^{0.65}\frac{d^{0.65}y(t)}{dt^{0.65}}+y(t)=e(t) \qquad H(s)=\frac{1}{1+(10s)^{0.65}}$$

N=10 and L=100





Relaxation Fractional Order System

Step response

$$Y(s)=\frac{1}{s[1+(s\tau_0)^m]}\cong\sum_{i=1}^{2N-1}\frac{G(\tau_i)}{s(1+s\tau_i)}=\sum_{i=1}^{2N-1}\frac{k_i}{s\left(1+\frac{s}{p_i}\right)}$$

$$Y(s)=\frac{1}{s[1+(s\tau_0)^m]}=\frac{1}{s}\left(\frac{\tau_0^{-m}}{s^m}\sum_{n=0}^{\infty}\frac{[-(\tau_0^{-m})]^n}{s^{nm}}\right)$$

Relaxation Fractional Order System

$$y(t) = \sum_{i=1}^{2N-1} k_i \left(1 - \exp(-t/\tau_i)\right)$$

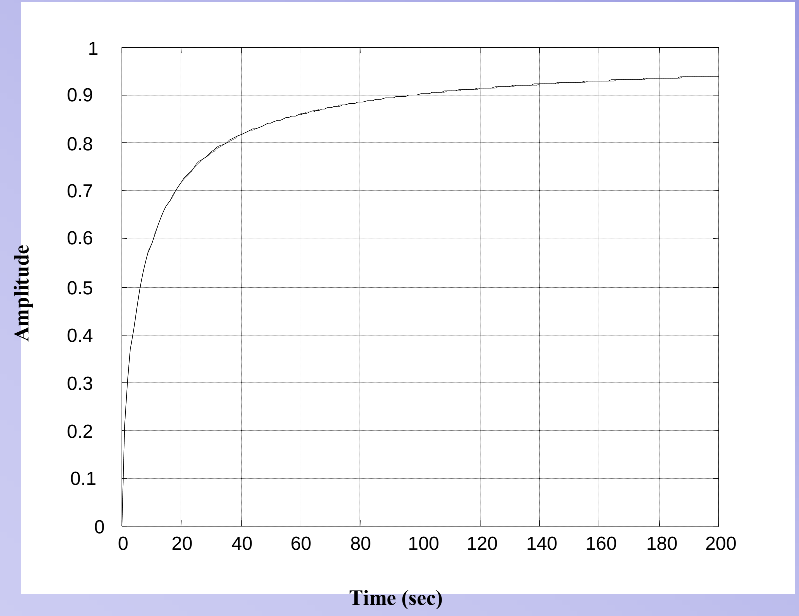
$$y(t) = 1 - \sum_{n=0}^{L} \frac{(-(\tau_0^{-m}))^n t^{nm}}{\Gamma(nm+1)}$$

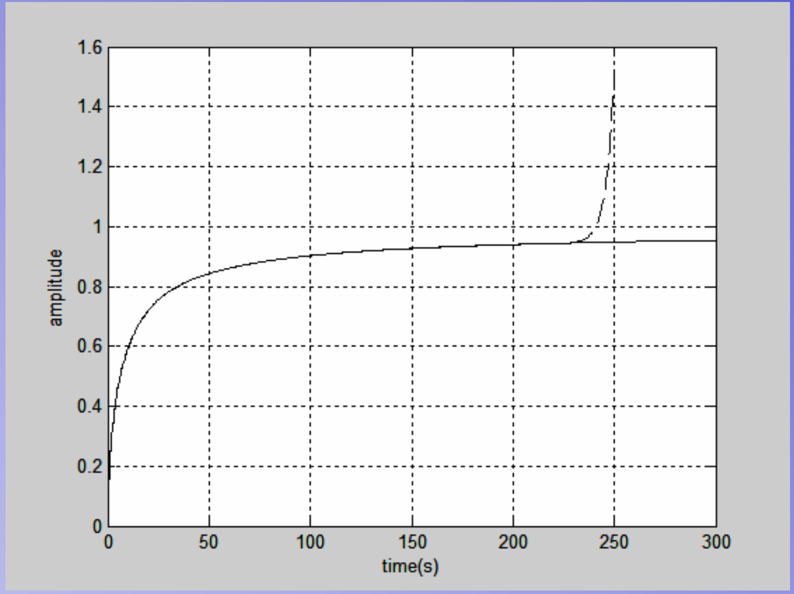
Relaxation Fractional Order System

Numerical example

$$(10)^{0.65}\frac{d^{0.65}y(t)}{dt^{0.65}}+y(t)=e(t) \qquad H(s)=\frac{1}{1+(10s)^{0.65}}$$

N=10 and L=100





Oscillation Fractional Order System

Oscillation Fractional Order System

$$(\tau_0)^m \frac{d^m x(t)}{dt^m} + x(t) = e(t) \quad , \qquad \text{for} \quad 1 < m < 2$$

Its corresponding transfer function

$$H(s) = \frac{X(s)}{E(s)} = \frac{1}{1+(s\tau_0)^m} \quad , \qquad \text{for} \quad 1 < m < 2$$

The expansion about s = ∞ using long division gives

$$H(s) = \frac{1}{1+(s\tau_0)^m} = \frac{\tau_0^{-m}}{s^m} \sum_{n=0}^{L} \frac{[-(\tau_0^{-m})]^n}{s^{nm}}$$

Oscillation Fractional Order System

$$H(s)=\frac{1}{1+\left(s\tau_0\right)^m}\approx\frac{\left(1+s\tau_0\right)^{2-m}}{\left(s\tau_0\right)^2+2\xi\left(s\tau_0\right)+1}$$ **, for 1 < m < 2**

With

$$\zeta=\sqrt{\frac{[1+\cos(\frac{\pi}{2}m)]}{2^{m-1}}}$$

And

$$\left(1+s\tau_0\right)^{2-m}=\frac{\prod_{i=0}^{N}(1+\frac{s}{z_i})}{\prod_{i=0}^{N}(1+\frac{s}{p_i})}$$

Because 0 < (2-m) < 1

Oscillation Fractional Order System

$$H(s) = \frac{1}{1+\left(s\tau_0\right)^m} \approx \frac{1}{\left(s\tau_0\right)^2 + 2\xi\left(s\tau_0\right) + 1} \frac{\prod_{i=0}^{N}\left(1+\frac{s}{z_i}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{p_i}\right)}$$

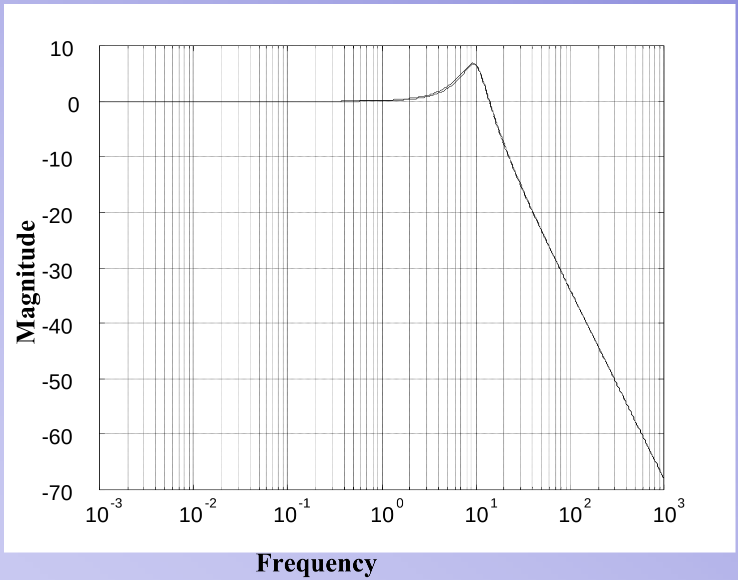
$$H(s) = \frac{As + B}{\left(\tau_0 s\right)^2 + 2\xi\left(\tau_0 s\right) + 1} + \sum_{n=0}^{N} \frac{k_i}{1+\frac{s}{p_i}}$$

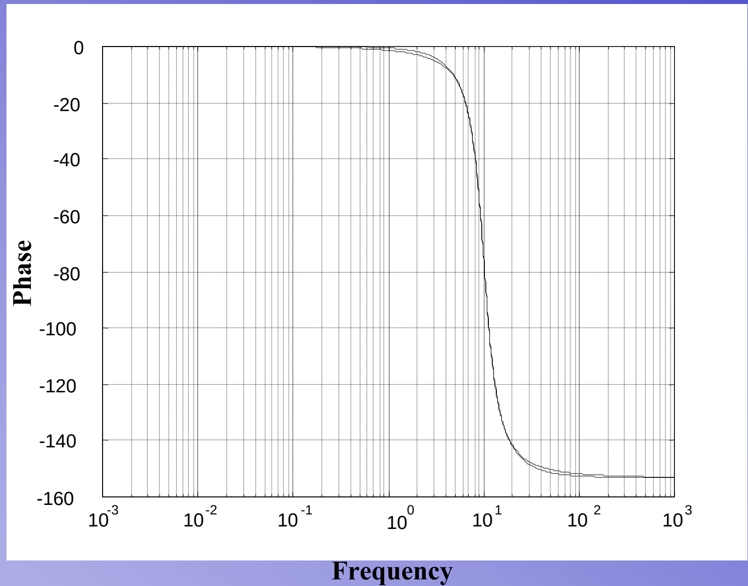
Oscillation Fractional Order System

Numerical example

$$(0.1)^{1.7}\frac{d^{1.7}x(t)}{dt^{1.7}}+x(t)=e(t)$$

$$\mathrm{H(s)}=\frac{1}{1+(0.1s)^{1.7}}$$





$$H(s)=\frac{As+B}{(\tau_0 s)^2+2\xi(\tau_0 s)+1}+\sum_{n=0}^{9}\frac{k_i}{1+\frac{s}{p_i}}$$

Oscillation Fractional Order System

$$H(s)=\frac{1}{1+(s\tau_0)^m}=\frac{\tau_0^{-m}}{s^m}\sum_{n=0}^{100}\frac{[-(\tau_0^{-m})]^n}{s^{nm}}$$

Oscillation Fractional Order System

Impulse response

$$h(t)=\tau_0\left(\frac{B^2-2AB\xi\tau_0^{-1}+A^2\tau_0^{-2}}{1-\xi^2}\right)^{1/2} e^{-\xi\tau_0^{-1}t}\sin(\tau_0^{-1}\sqrt{(1-\xi^2)}t+\varphi)$$

$$+\sum_{i=0}^{N} k_i\, p_i\, e^{-p_i t}$$

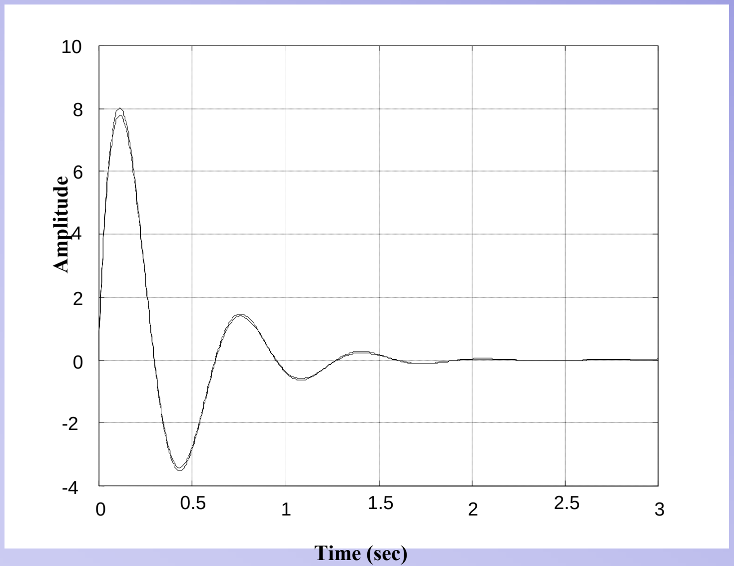
$$h(t)=(\tau_0^{-m})t^{(m-1)}\sum_{n=0}^{L}\frac{(-(\tau_0^{-m}))^n t^{nm}}{\Gamma(nm+m)}$$

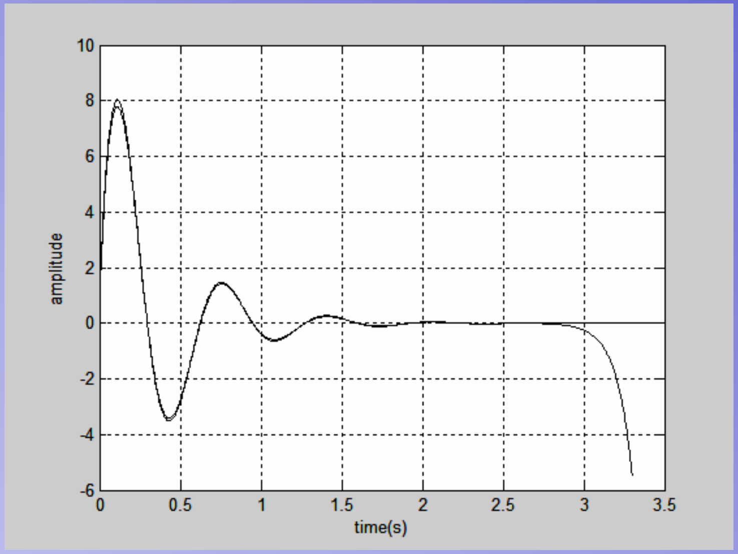
Oscillation Fractional Order System

Numerical example

$$(0.1)^{1.7}\frac{d^{1.7}x(t)}{dt^{1.7}}+x(t)=e(t) \qquad H(s)=\frac{1}{1+(0.1s)^{1.7}}$$

N=9 and L=100





Oscillation Fractional Order System

Step response

$$Y(s)=\frac{1}{s[1+(s\tau_0)^m]}=\frac{1}{s}\frac{As+B}{(\tau_0 s)^2+2\xi(\tau_0 s)+1}+\sum_{n=0}^{N}\frac{1}{s}\frac{k_i}{1+\frac{s}{p_i}}$$

$$Y(s)=\frac{1}{s[1+(s\tau_0)^m]}=\frac{1}{s}\left(\frac{\tau_0^{-m}}{s^m}\sum_{n=0}^{\infty}\frac{[-(\tau_0^{-m})]^n}{s^{nm}}\right)$$

Oscillation Fractional Order System

$$y(t) = 1 + \tau_0 \left(\frac{B^2 - 2AB\xi\tau_0^{-1} + A^2\tau_0^{-2}}{1-\xi^2} \right)^{1/2} e^{-\xi\tau_0^{-1}t} \sin(\tau_0^{-1}\sqrt{(1-\xi^2)}t + \varphi_1)$$

$$-\sum_{i=0}^{N} k_i\, p_i\, e^{-p_i t}$$

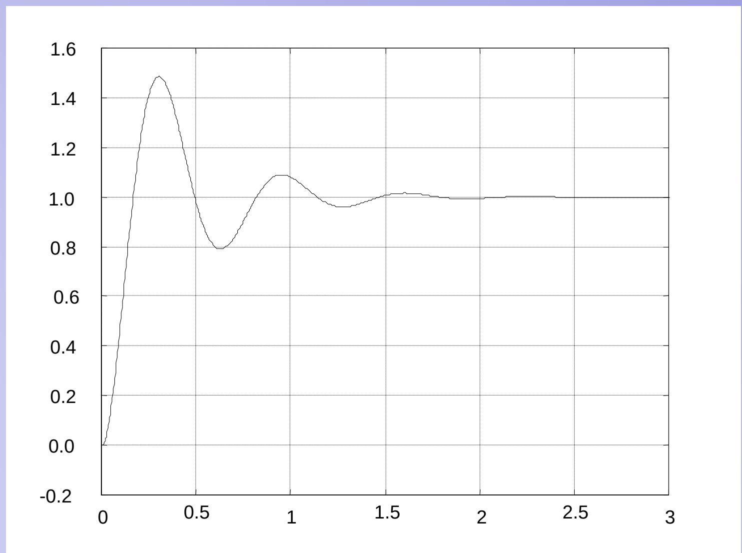
$$y(t) = 1 - \sum_{n=0}^{L} \frac{(-(\tau_0^{-m}))^n t^{nm}}{\Gamma(nm+1)}$$

Oscillation Fractional Order System

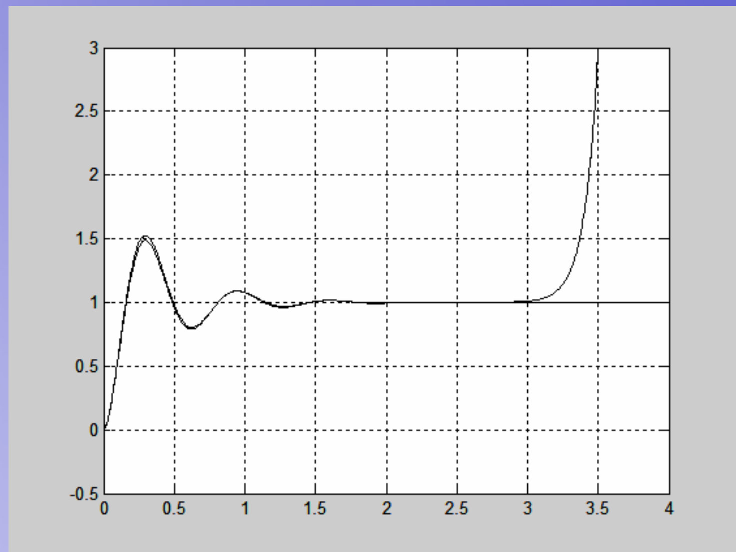
Numerical example

$$(0.1)^{1.7}\frac{d^{1.7}x(t)}{dt^{1.7}}+x(t)=e(t) \qquad H(s)=\frac{1}{1+(0.1s)^{1.7}}$$

N=9 and L=100



Time (sec)



Conclusion

An effective methods for approximating the irrational transfer function the fundamental linear fractional order differential equation by a rational function, in a given frequency band is derived.

The impulse and step responses of this type of systems are derived.

Illustrative examples have been treated to demonstrate the usefulness of the approximation methods.

Conclusion

Theses approximations can very suitable for analysis, realization and implementation of fractional order systems.

The expressions for characteristics and usual time and frequency specifications can also be derived.

The approximation presented is better than the expansion about $s = \infty$ using long division which is shown to diverge in the time domain.

Conclusion

Thank you for your attention